

Local Properties and Differential Forms of Smooth Map and Tangent Bundle

Md. Shafiul Alam[1], Khondokar M. Ahmed[2] & Bijan Krishna Saha[3]

[1]Department of Mathematics, University of Barisal, Barisal, Bangladesh

[2]Department of Mathematics, University of Dhaka, Dhaka, Bangladesh

[3]Department of Mathematics, University of Barisal, Barisal, Bangladesh

[1]Corresponding author: Email: shafiulmt@gmail.com, Tel: +88-01819357123

Abstract

In this paper, some basic properties of differential forms of smooth map and tangent bundle are developed. The linear map $T_a(M) \to T_{\varphi(a)}(N)$ defined by $\xi \mapsto \xi \circ \varphi^*$ is the derivative of φ at a, where $\varphi: M \to N$ is a smooth map and $a \in M$. If $\varphi: M \to N$ is a smooth bijective map and if the maps $(d\varphi)_x: T_x(M) \to T_{\varphi(x)}(N)$ are all injective, then φ is a diffeomorphism. Finally, it is shown that if X is a vector field on a manifold M then there is a radial neighbourhood W of $0 \times M$ in $\mathbb{R} \times M$ and a smooth map $\varphi: W \to M$ such that $\dot{\varphi}_x(t) = X(\varphi_x(t)), (t,x) \in W$ and $\varphi_x(0) = x, x \in M$, where φ_x is given by $\varphi_x(t) = \varphi(t,x)$.

Keywords: Smooth map, manifold, tangent bundle, isomorphism, diffeomorphism, vector field.

1. Introduction

S. Kobayashi[1] and K. Nomizu[1] first worked on differential forms of smooth map and tangent bundle. Later H. Flanders[2], F. Warner[3], M. Spivak[4] and J. W. Milnor[5] also worked on differential forms of smooth map and tangent bundle. A number of significant results were obtained by S. Cairns[6], W. Greub[9], E. Stamm[9], R. G. Swan[10], R. L. Bishop[12], R. J. Crittenden[12] and others.

Let M be a smooth manifold and let $\mathscr{S}(M)$ be the ring of smooth functions on M. A tangent vector of M at a point $a \in M$ is a linear map $\xi: \mathscr{S}(M) \to \mathbb{R}$ such that

$$\xi(fg) = \xi(f)g(a) + f(a)\,\xi(g), \quad f,g \in \mathscr{S}(M).$$

The tangent vectors form a real vector space $T_a(M)$ under the linear operations

$$(\lambda\xi + \mu\eta)(f) = \lambda\xi(f) + \mu\eta(f), \quad \lambda,\mu \in \mathbb{R}, \quad \xi,\eta \in T_a(M), \quad f \in \mathscr{S}(M).$$

$T_a(M)$ is called the tangent space of M at a.

Let E be an n-dimensional real vector space. Let O be an open subset of E and let $a \in O$. We shall define a linear isomorphism $\lambda_a: E \xrightarrow{\cong} T_a(O)$. If $\varphi: O \to F$ is a smooth map of O into a second vector space F, then the classical derivative of φ at a is the linear map $\varphi'(a): E \to F$ given by

$$\varphi'(a;h) = \lim_{t\to 0} \frac{\varphi(a+th) - \varphi(t)}{t}, \quad h \in E.$$

Moreover, in the special case $F = \mathbb{R}$ we have the product formula

$$(fg)'(a;h) = f'(a;h)\,g(a) + f(a)\,g'(a;h), \quad f,g \in \mathscr{S}(O).$$

This shows that the linear map $\xi_h: \mathscr{S}(O) \to \mathbb{R}$ given by $\xi_h(f) = f'(a;h)$ is a tangent vector of O at a. Hence, we have a canonical linear map $\lambda_a: E \to T_a(O)$ given by $\lambda_a: h \to \xi_h\,,\ h \in E$.

2. The Derivative of a Smooth Map

Let $\varphi: M \to N$ be a smooth map and let $a \in M$. The linear map $T_a(M) \to T_{\varphi(a)}(N)$ defined by $\xi \mapsto \xi \circ \varphi^*$ is called the derivative of φ at a. It will be denoted by $(d\varphi)_a$,

$$((d\varphi)_a\xi)(g) = \xi(\varphi^* g),\ \ g \in \mathscr{S}(N),\ \ \xi \in T_a(M).$$

If $\psi: N \to Q$ is a second smooth map, then $(d(\psi \circ \varphi))_a = (d\psi)_{\varphi(a)} \circ (d\varphi)_a$, $a \in M$. Moreover, for the identity map $i: M \to M$ we have $(di)_a = i_{T_a(M)}$, $a \in M$. In particular, if $\varphi: M \to N$ is a diffeomorphism, then $(d\varphi)_a: T_a(M) \to T_{\varphi(a)}(N)$ and $(d\varphi^{-1})_{\varphi(a)}: T_{\varphi(a)}(N) \to T_a(M)$ are inverse linear isomorphisms.

Lemma 1. Let $\xi \in T_a(M)$, then $\xi \circ \varphi^* \in T_{\varphi(a)}(N)$ and the correspondence $\xi \mapsto \xi \circ \varphi^*$ defines a linear map from $T_a(M)$ to $T_{\varphi(a)}(N)$.

Proof. Let $\varphi: M \to N$ be a smooth map. φ induces a homomorphism $\varphi^*: \mathscr{S}(N) \to \mathscr{S}(M)$ given by $(\varphi^* f)(x) = f(\varphi(x))$, $f \in \mathscr{S}(N)$, $x \in M$.

$\xi \circ \varphi^*$ is a linear map from $\mathscr{S}(N)$ to $\mathbb{R}$. Moreover,

$$(\xi \circ \varphi^*)(fg) = \xi(\varphi^* f \cdot \varphi^* g) = \xi(\varphi^* f) \cdot g(\varphi(a)) + f(\varphi(a)) \cdot \xi(\varphi^* g),\ \ f, g \in \mathscr{S}(N)$$

and so $\xi \circ \varphi^* \in T_{\varphi(a)}(N)$. Clearly $\xi \mapsto \xi \circ \varphi^*$ is linear. Therefore, the lemma is proved.

Proposition 1. Let $e_i\,(i = 1, \cdots, n)$ be a basis for E and let $f \in \mathscr{S}(E)$. Then

$$f = f(a) + \sum_{i=1}^{n} h_i g_i$$

(a) the functions $h_i \in \mathscr{S}(E)$ are given by $x - a = \sum_{i=1}^{n} h_i(x) e_i$, $x \in E$,

(b) the functions $g_i \in \mathscr{S}(E)$ satisfy $g_i(a) = f'(a; e_i)$, $i = 1, \cdots, n$.

Proof. By the fundamental theorem of calculus we have

$$f(x) - f(a) = \int_0^1 \frac{d}{dt} f(a + t(x - a))dt$$

$$= \int_0^1 f'^{(a+t(x-a);x-a)} dt$$

$$= \sum_{i=1}^{n} h_i(x) \int_0^1 f'(a + t(x - a); x - a)\, dt.$$

Thus the theorem follows, with $g_i(x) = \int_0^1 f'(a + t(x - a);\, e_i)dt$, $x \in E$.

Proposition 2. The canonical linear map $\lambda_a: E \to T_a(O)$ is an isomorphism.

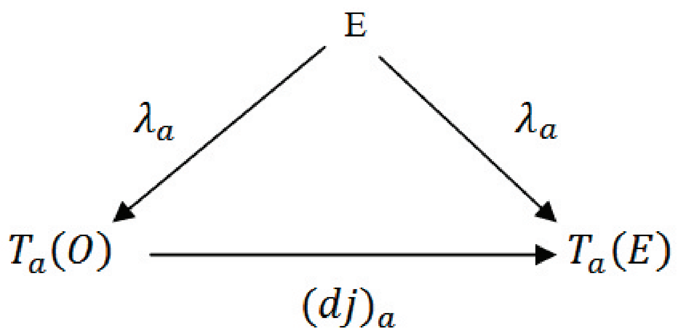
Proof. First we consider the case $O = E$. We show first that λ_a is surjective. Let $\xi \in T_a(E)$ and let $f \in \mathscr{S}(E)$. Write $f = f(a) + \sum_{i=1}^{n} h_i g_i$, where the h_i, g_i satisfy conditions (a) and (b) of Proposition 1. Since ξ maps the constant function $f(a)$ into zero, so

$$\xi(f) = \sum_{i=1}^{n} \xi(h_i)\, g_i(a) + \sum_{i=1}^{n} h_i(a)\, \xi(g_i)$$

$$= \sum_{i=1}^{n} \xi(h_i)\, f'(a;\, e_i) = \left(\sum_{i=1}^{n} \xi(h_i)\, \xi_{e_i} \right)(f).$$

Since the functions h_i are independent of f, we can write $\xi = \sum_{i=1}^{n} \xi(h_i) \xi_{e_i} = \lambda_a(\sum_{i=1}^{n} \xi(h_i)\, e_i)$. Thus λ_a is surjective. To show that λ_a is injective, let f be any linear function in E. Then for $h \in E$, $\lambda_a(h)(f) = f(h)$. Now suppose $\lambda_a(h) = 0$. Then $f(h) = 0, f \in E^*$. Hence $h = 0$ and so λ_a is injective.

Finally, let O be any open subset of E and let $j: O \to E$ be the inclusion map. Then $j'(a): E \to E$ is the identity map and we have the commutative diagram



Since $(dj)_a$ is a linear isomorphism, the canonical linear map $\lambda_a: E \to T_a(O)$ is an isomorphism, which completes the proof.

Corollary 1. Let M be a smooth manifold and let $a \in M$. Then $\dim T_a(M) = \dim M$.

Proof. Let $(U, u, \widehat{U})$ be a chart for M such that $a \in U$. Using the result of Lemma 1 and Proposition 2 we find $\dim T_a(M) = \dim T_a(U) = \dim U = \dim M$.

Proposition 3. The derivative of a constant map is zero. Conversely, let $\varphi: M \to N$ be a smooth map such that $(d\varphi)_a = 0,\ a \in M$. Then, if M is connected, φ is a constant map.

Proof. Assume that φ is the constant map $M \to b$, where $b \in N$. Then, for $g \in \mathscr{S}(N)$ is the constant function given by $(\varphi^* g)(x) = g(b),\ x \in M$. Hence, for $\xi \in T_a(M), a \in M$, $(d\varphi)_a(\xi)(g) = \xi(\varphi^* g) = 0$. It follows that each $(d\varphi)_a = 0$.

Conversely, assume that $\varphi: M \to N$ is a smooth map satisfying $(d\varphi)_a = 0, a \in M$ and let M be connected. Then, given two points $x_0 \in M$ and x_1, there exists a smooth curve $f: \mathbb{R} \to M$ such that $f(0) = x_0$ and $f(1) = x_1$. Consider the map $g = \varphi \circ f: \mathbb{R} \to N$. We have $(dg)_t = (d\varphi)_{f(t)} \circ (df)_t = 0,\ t \in \mathbb{R}$.

Now using an atlas for N we see that g must be constant. In particular $g(0) = g(1)$ and so $\varphi(x_0) = g(0) = g(1) = \varphi(x_1)$. Thus φ is a constant map and the proposition is proved.

3. Local Properties of Smooth Maps

Let $\varphi: M \to N$ be a smooth map. Then φ is called a local diffeomorphism at a point $a \in M$ if the map $(d\varphi)_a: T_a(M) \to T_{\varphi(a)}(N)$ is a linear isomorphism. If φ is a local diffeomorphism for all points $a \in M$, it is called a local diffeomorphism of M into N.

Theorem 1. Let $\varphi: M \to N$ be a smooth map where $\dim M = n$ and $\dim N = r$. Let $a \in M$ be a given point. Then

(a) If φ is a local diffeomorphism at a, there are neighbourhoods U of a and V of b such that φ maps U diffeomorphically onto V.

(b) If $(d\varphi)_a$ is injective, there are neighbourhoods U of a, V of b, and W of 0 in $\mathbb{R}^{r-n}$ and a diffeomorphism $\psi: U \times W \xrightarrow{\cong} V$ such that $\varphi(x) = \psi(x, 0), x \in U$.

(c) If $(d\varphi)_a$ is surjective, there are neighbourhoods U of a, V of b, and W of 0 in $\mathbb{R}^{r-n}$, and a diffeomorphism $\psi: U \xrightarrow{\cong} V \times W$ such that $\varphi(x) = \pi_V\, \psi(x),\ x \in U$ where $\pi_V: V \times W \to V$ is the projection.

Proof. By using charts we may reduce to the case $M = \mathbb{R}^n, N = \mathbb{R}^r$. In part (a), then, we are assuming that $\varphi'(a): \mathbb{R}^n \to \mathbb{R}^r$ is an isomorphism, and the conclusion is the inverse function theorem.

For part (b), we choose a subspace E of $\mathbb{R}^r$ such that $Im\, \varphi'(a) \oplus E = \mathbb{R}^r$, and consider the map $\psi: \mathbb{R}^n \times E \to \mathbb{R}^r$ given by $\psi(x, y) = \varphi(x) + y,\ x \in \mathbb{R}^n, y \in E$. Then

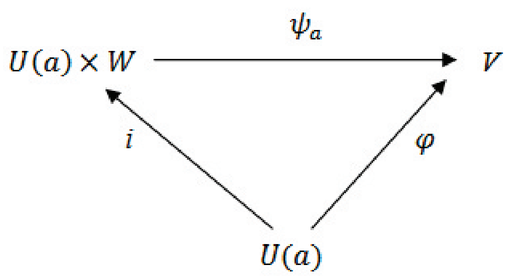
$$\psi'(a,0;h,k) = \varphi'(a;h) + k, h \in \mathbb{R}^n,\ k \in E.$$

It follows that $\psi'(a,0)$ is injective and thus an isomorphism (r = dim Im $\varphi'(a)$ + dim E = n + dim E). Thus part (a) implies the existence of neighborhoods U of a, V of b, and W of 0 in E such that $\psi: U \times W \ \to V$ is a diffeomorphism. Clearly, $\psi(x,0) = \ \varphi(x)$.

Finally, for part (c), we choose a subspace E of $\mathbb{R}^n$ such that $\ker\varphi'(a) \oplus E = \mathbb{R}^n$. Let $\rho: \mathbb{R}^n \to E$ be the projection induced by this decomposition and define $\psi: \mathbb{R}^n \to \mathbb{R}^r \oplus E$ by $\psi(x) = \big(\varphi(x), \rho(x)\big)$, $x \in \mathbb{R}^n$. Then $\psi'^{(a;h)} = \left(\varphi'^{(a;h)}, \rho(h)\right)$, $a \in \mathbb{R}^n$, $h \in \mathbb{R}^n$. It follows easily that $\psi'(a)$ is a linear isomorphism. Hence there are neighborhoods U of a, V of b and W of $0 \in E$ such that $\psi: U \to V \times W$ is a diffeomorphism. Hence the proof of the theorem is complete.

Proposition 4. If $\varphi: M \to N$ is a smooth bijective map and if the maps $(d\varphi)_x: T_x(M) \to T_{\varphi(x)}(N)$ are all injective, then φ is a diffeomorphism.

Proof. Let dim M = n, dim N = r. Since $(d\varphi)_x$ is injective, we have $r \geq n$. Now we show that r = n. In fact, according to Theorem 1, part (b), for every $a \in M$ there are neighborhoods $U(a)$ of a, V of $\varphi(a)$ and W of $0 \in \mathbb{R}^{r-n}$ together with a diffeomorphism $\psi_a: U(a) \times W \xrightarrow{\cong} V$ such that the diagram



commutes (i denotes the inclusion map opposite 0).

Choose a countable open covering U_i $(i = 1, 2, \cdots)$ of M such that each $\overline{U}_i$ is compact and contained in some $U(a_i)$. Since φ is surjective, it follows that $N \subset \bigcup_i \varphi(\overline{U}_i)$. Now assume that r > n. Then the diagram implies that no $\varphi(\overline{U}_i)$ contains an open set. Thus, N could not be Hausdorff. This contradiction shows that $n = r$. Since $n = r$, φ is a local diffeomorphism. On the other hand, φ is bijective. Since it is a local diffeomorphism, Theorem 1 implies that its inverse is smooth. Thus φ is a diffeomorphism and the theorem is proved.

Definition 1. A quotient manifold of a manifold M is a manifold N together with a smooth map $\pi: M \to N$ such that π and each linear map $(d\pi)_x: T_x(M) \to T_{\pi(x)}(N)$ is surjective and thus dim $M \geq$ dim N.

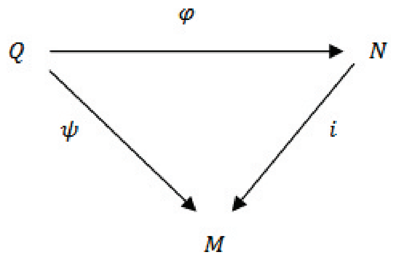
Lemma 2. Let $\pi: M \to N$ make N into a quotient manifold of M. Then the map π is open.

Proof. It is sufficient to show that, for each $a \in M$, there is a neighbourhood U of a such that the restriction of π to U is open. This follows at once from part (c) of Theorem 1.

4. Submanifolds

Let M be a manifold. An embedded manifold is a pair (N, φ), where N is a second manifold and $\varphi: N \ \to M$ is *a* smooth map such that the derivative $d\varphi: T_N \to T_M$ is injective. In particular, since the maps $(d\varphi)_x: T_x(N) \to \ T_{\varphi(x)}(M)$ are injective, it follows that dim $N \leq$ dim M. Given an embedded manifold (N, φ), the subset $M_1 = \varphi(N)$ may be considered as a bijective map $\varphi_1: N \to M_1$. A submanifold of a manifold M is an embedded manifold (N, φ) such that $\varphi_1: N \ \to \ \varphi(N)$ is a homeomorphism, when $\varphi(N)$ is given the topology induced by the topology of M. If N is a subset of M and φ is the inclusion map, we say simply that N is a submanifold of M.

Proposition 5. Let (N, i) be a submanifold of M. Assume that Q is a smooth manifold and



is a commutative diagram of maps. Then φ is smooth if and only if ψ is smooth.

Proof. If φ is smooth then clearly ψ is smooth. Conversely, assume that ψ is smooth. Fix a point $a \in Q$ and set $b = \psi(a)$. Since di is injective, there are neighbourhoods U, V of b in N and M respectively, and there is a smooth map $\lambda: V \to U$ such that $\lambda \circ i_U = \iota$.

Since N is a submanifold of M, the map φ is continuous. Hence there is a neighbourhood W of a such that $\varphi(W) \subset U$. Then $i_U \circ \varphi_W = \psi_W$, where φ_W, ψ_W denote the restrictions of φ, ψ to W. It follows that $\lambda \circ \psi_W = \lambda \circ j_U \circ \varphi_W = \varphi_W$ and so φ is smooth in W; thus φ is a smooth map. Hence the proposition is proved.

5. Vector Fields

A vector field X on a manifold M is a cross-section in the tangent bundle T_M. Thus a vector field X assigns to every point $x \in M$ a tangent vector $X(x)$ such that the map $M \to T_M$ so obtained is smooth. The vector fields on M form a module over the ring $\mathscr{S}(M)$ which will be denoted by $\mathcal{X}(M)$.

Theorem 2. There is a canonical isomorphism of $\mathcal{X}(M)$ onto the $\mathscr{S}(M)$-module of derivations in the algebra $\mathscr{S}(M)$, i.e., Der $\mathscr{S}(M)$.

Proof. Let X be a vector field. For each $f \in \mathscr{S}(M)$, we define a function $X(f)$ on M by $X(f)(x) = X(x)(f)$, $x \in M$. $X(f)$ is smooth. To see this we may assume that $M = \mathbb{R}^n$. But then $X(f)(x) = f'(x; X(x))$ is smooth. Hence every vector field X on M determines a map

$$\theta_X: \mathscr{S}(M) \to \mathscr{S}(M) \text{ given by } \theta_X(f) = X(f).$$

Obviously, θ_X is a derivation in the algebra $\mathscr{S}(M)$. The assignment $X \mapsto \theta_X$ defines a homomorphism $\theta: \mathcal{X}(M) \to \text{Der } \mathscr{S}(M)$.

We show now that θ is an isomorphism. Suppose $\theta_X = 0$, for some $X \in \mathcal{X}(M)$. Then

$$X(x)f = 0, \; x \in M, \; f \in \mathscr{S}(M).$$

This implies that $X(x) = 0$; i.e. $X = 0$. To prove that θ is surjective, let Φ be any derivation in $\mathscr{S}(M)$. Then, for every point $x \in M$, Φ determines the vector $\xi_x \in T_x(M)$, given by

$$\xi_x(f) = \Phi(f)(x), f \in \mathscr{S}(M).$$

We define $X: M \to T_M$ by $X(x) = \xi_x$. To show that this map is smooth, fix a point $a \in M$. Using a chart, it is easy to construct vector fields X_i $(i = 1, \cdots, n)$ and smooth functions f_i $(i = 1, \cdots, n;\ n = \dim M)$ on M such that $X_i(x)f_j = \delta_{ij}$, $x \in V$, (V is some neighbourhood of a). Then the vectors $X_i(x)$ $(i = 1, \cdots, n)$ form a basis for $T_x(M)$ $(x \in V)$. Hence, for each $x \in V$, there is a unique system of numbers λ_x^i $(i = 1, \cdots, n)$ such that $\xi_x = \sum_{i=1}^{n} \lambda_x^i X_i(x)$. Applying ξ_x to f_i we obtain $\lambda_x^i = \xi_x(f_i) = \Phi(f_i)(x)$, $x \in V$. Hence $X(x) = \sum_{i=1}^{n} \Phi(f_i)(x) X_i(x)$, $x \in V$. Since the $\Phi(f_i)$ are smooth functions on M, this equation shows that X is smooth in V; i.e. X is a vector field. Finally, it follows from the definition that $\theta_X = \Phi$. Thus θ is surjective, which completes the proof.

6. Differential Equations

Let X be a vector field on a manifold M. An orbit for X is a smooth map $\alpha: I \to M$ where $I \subset \mathbb{R}$ is some open interval such that $\dot{\alpha}(t) = X(\alpha(t)),\ t \in I$.

Consider the product manifold $\mathbb{R} \times M$. We call a subset $W \subset \mathbb{R} \times M$ radial if for each $a \in M$, $W \cap (\mathbb{R} \times a) = I_a \times a$ or $W \cap (\mathbb{R} \times a) = \emptyset$ where I_a is an open interval on $\mathbb{R}$ containing the point 0. The union and finite intersection of radial sets is again radial.

Proposition 6. Let X be a vector field on M. Fix $a \in M$ and $t_0 \in \mathbb{R}$. Then there is an interval I and an orbit $\alpha: I \to M$ of X such that $\alpha(t_0) = a$. Moreover, if $\alpha, \beta: J \to M$ are orbits for X which agree at some $s_0 \in J$, then $\alpha = \beta$.

Proof. For the first statement of the proposition we may assume $M = \mathbb{R}^n$. In this case it is the standard Picard existence theorem.

To prove the second part we show that the set of $s \in J$ for which $\alpha(s) = \beta(s)$ is both closed and open, and hence all of J. It is obviously closed. To show that it is open we may assume $M = \mathbb{R}^n$ and then we apply the Picard uniqueness theorem, which completes the proof.

Theorem 3. Let X be a vector field on a manifold M. Then there is a radial neighbourhood W of $0 \times M$ in $\mathbb{R} \times M$ and a smooth map $\varphi: W \to M$ such that

$$\dot{\varphi}_x(t) = X(\varphi_x(t)),\quad (t, x) \in W \quad \text{and} \quad \varphi_x(0) = x,\quad x \in M,$$

where φ_x is given by $\varphi_x(t) = \varphi(t, x)$. Moreover, φ is uniquely determined by X.

Proof. Let $\{(U_\alpha, u_\alpha)\}$ be an atlas for M. The Picard existence theorem implies our theorem for each U_α. Hence there are radial neighbourhoods W_α of $0 \times U_\alpha$ in $\mathbb{R} \times U_\alpha$ and there are smooth maps φ_α such that $\dot{\varphi}_\alpha(t, x) = X(\varphi_\alpha(t)),\ (t, x) \in W_\alpha$ and $\varphi_\alpha(0, x) = x,\ x \in U_\alpha$.

Now set $W = \bigcup_\alpha W_\alpha$. Then W is a radial neighbourhood of $0 \times M$ in $\mathbb{R} \times M$. Moreover, $W_\alpha \cap W_\beta$ is a radial neighbourhood of $0 \times (U_\alpha \cap U_\beta)$; if $x \in U_\alpha \cap U_\beta$, then $(W_\alpha \cap W_\beta) \cap (\mathbb{R} \times x)$ is an interval I containing 0. Clearly, $\varphi_\alpha, \varphi_\beta: I \times x \to M$ are orbits of X agreeing at 0, and so by Proposition 6 they agree in I. It follows that they agree in $W_\alpha \cap W_\beta$. Thus the φ_α defines a smooth map $\varphi: W \to M$ which has the desired properties. The uniqueness of φ follows from Proposition 6. Thus the theorem is proved.

7. Conclusions

Differential forms are among the fundamental analytic objects treated in this paper. The derivative of a smooth map appears as a bundle map between the corresponding tangent bundles. They are the cross-sections in the exterior algebra bundle of the dual of the tangent bundle and they form a graded anticommutative algebra. Vector fields on a manifold has been introduced as cross-sections in the tangent bundle. The module of vector fields is canonically isomorphic to the module of derivations in the ring of smooth functions.

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