A cubic spline collocation method for solving Bratu's Problem

Abdelmajid El hajaji [1*], Khalid Hilal [1], El merzguioui Mhamed [2] and El ghordaf Jalila [1]
[1] Department of Mathematics, Faculty of Science and Technology,
University Sultan Moulay Slimane, Beni-Mellal, Morocco
[2] Department of Mathematics, Faculty of Science and Technology,
University Abdelmalek Essaadi, Tanger, Morocco
[*]E-mail: a_elhajaji@yahoo.fr

Abstract

This paper, we develop a numerical method for solving a Bratu-type equations by using the cubic spline collocation method (CSCM) and the generalized Newton method. This method converges quadratically if a relation-ship between the physical parameter λ and the discretization parameter h is satisfied. An error estimate between the exact solution and the discret solution is provided. To validate the theoretical results, Numerical results are presented and compared with other collocation methods given in the literature.

Keywords: Bratu-type equations, Boundary value problems, Cubic spline collocation method.

1. Introduction

The Bratu problem appears in a large variety of applications such as: Fuel ignition model of thermal combustion, radiative heat transfer, thermal reaction, the Chandrasekhar model of the expansion of the universe, chemical reactor theory and nanotechnology [15, 16]. In [16] a summary of the history of the problem is given.

Studies on fuel ignition in thermal combustion theory have been on the increase over the last few years. The reason for the increased study is to ensure the safety of working environment especially when working with combustible fluid in some petro-chemical engineering processes. Combustion problems are generally characterized by strong nonlinearity and singularity, as such in most cases exact solution of combustion problems are very difficult to get. Therefore, researchers working in this area have resolved to approximate solutions by either analytical or numerical method.

Nonlinear phenomena are of fundamental importance in various fields of science and engineering. The nonlinear models of real-life problems are still difficult to solve either numerically or analytically. There has recently been much attention devoted to the search for better and more efficient solution methods for determining a solution, approximate or exact, analytical or numerical, to nonlinear models, [20, 14, 19].

Consider the Liouville-Bratu-Gelfand equation [15]

$$\begin{cases} -\Delta u &= \lambda e^{u} \quad \text{on } \Omega, \\ \quad u &= 0, \quad \text{on } \partial\Omega \end{cases}$$

where $\lambda > 0$, and Ω is a bounded domain. We consider the classical Bratu's problem [16] given by the following boundary-value problem

$$\begin{cases} -\Delta u = \lambda e^{u} & \text{on}(0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

which is used to model a combustion problem in a numerical slab.

Several numerical techniques, like the variational iteration method (VIM) [9], non-polynomial spline method [7], the Adomain decomposition method [3], Chebyshev wavelets method [5, 8], the parametric cubic spline method [6] and the Laplace transform decomposition method [4] have been implemented independently to handle the Bratu model numerically.

In this paper we develop a numerical method for solving a one dimentional Bratu's problem by using the CSCM and the generalized Newton method. First, we apply the spline collocation method to approximate the solution of a boundary value problem of second order. The discret problem is formulated as to find the cubic spline coefficients of a nonsmooth system $\varphi(Y) = Y$, where $\varphi : \mathbb{R}^m \to \mathbb{R}^m$. In order to solve the nonsmooth equation we apply the generalized Newton method (see [10, 11, 12], for instance). We prove that the CSCM converges quadratically provided that a property coupling the parameter λ and the discretization parameter h is satisfied.
Numerical methods to approximate the solution of boundary value problems have been considered by several authors. We only mention the papers [13, 14] and references therein, which use the spline collocation method for solving the boundary value problems.

The cubic B-spline collocation method is widely used in practice because it is computationally inexpensive, easy to implement and gives high-order accuracy. In [18, 19] the authors solved a the Bratu's problem by using third degree splines, where they considerer the collocation points as the knots of the cubic spline space. In our paper we consider a cubic spline space defined by multiple knots in the boundary and we propose a simple and efficient new collocation method by considering as collocation points the mid-points of the knots of the cubic spline space. It is observed that the collocation method developed in this paper, when applied to some examples, can improve the results obtained by the collocation methods given in the literature (see [18, 19, 20, 21, 22, 23, 24], for instance).

The present paper is organized as follows. In Section 2, we present the Bratu's Problem. In Section 3, we construct a cubic spline to approximate the solution of the boundary problem and we present the generalized Newton method. In Section 4 we show the convergence of the cubic spline to the solution of the boundary problem and provide an error estimate. The obtained numerical results are compared to the [18, 19, 20, 21, 22, 23, 24] in Section 5. Finally, a conclusion is given in Section 6.
The exact solution of the Bratu's problem (1) is given by

$$u(x) = -2\ln\left\{\frac{\cosh[(x-1/2)\theta/2]}{\cosh(\theta/4)}\right\},$$

in which θ is the solution of the equation $\theta = \sqrt{2\lambda}\cosh(\theta/4)$. This Bratu problem has zero, one or two solutions associated with $\lambda > \lambda_c$, $\lambda < \lambda_c$, respectively, where $\lambda_c = 3.513830719$ [19–20] satisfies the equation $4 = \sqrt{2\lambda_c}\sinh(\theta_c/4)$.

In this paper, we shall apply CSCM to find the approximate analytical solution of the boundary and initial value problem of the Bratu-type model. Comparisons with the exact solution shall be performed.

2. Bratu's Problem

Applications of the Bratu type equation are employed in the fuel ignition Model of the thermal combustion theory, the model of thermal reaction process, the Chandrasekhar model of the expansion of the universe, questions in geometry and relativity about the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology [7]

In this paper, we consider the Bratu's boundary value problem in one-dimensional planar coordinates in the form

$$\begin{cases} -\Delta u &= J(\cdot, u) \quad \text{on } (0,1), \\ u(0) &= u(1) = 0, \end{cases} \tag{2}$$

with $J = \lambda e^u$ where the physical parameter $\lambda > 0$.
It is easy to see that J is a nonlinear continuous function on u; and for any two functions u and v, J satisfies the following Lipschitz condition:

$$|J(x,u(x)) - J(x,v(x))| \le \lambda |u(x) - v(x)| \quad a.e. \; on \; x \in I, \tag{3}$$

Now, we define the following interpolation cubic spline of the solution u of the nonlinear second order boundary value problem (6).

3. Cubic spline solution

In this section we construct a cubic spline which approximates the solution u of problem (3), with I is the interval $I = (a,b) \subset \mathbb{R}$.

Let $\tau = \{a = x_{-3} = x_{-2} = x_{-1} = x_0 < x_1 < \cdots < x_{n-1} < x_n = x_{n+1} = x_{n+2} = x_{n+3} = b\}$ be a subdivision of the interval I. Without loss of generality, we put $x_i = a + ih$, where $0 \le i \le n$ and $h = (b-a)/n$. Denote by $\overset{\circ}{S}_4(I,\tau) = \{s \in C^2(I) \cap C^0(\bar{I}), s(a) = s(b) = 0, s_{(x_i, x_{i+1})} \in P(x_i, x_{i+1})\}$ the space of piecewise polynomials of degree 3 over the subdivision τ and of class C^2 everywhere on I and class C^0 everywhere on $\bar{I}$. Note that $\overset{\circ}{S}_4(I,\tau) \subseteq \overset{\circ}{H}{}^1(I)$. Let B_i, $i = -3, \cdots, n-1$, be the B-splines of degree 3 associated with τ. These B-splines are positives and form a basis of the space $\overset{\circ}{S}_4(I,\tau)$.

Proposition 3.1: *Let u be the solution of problem (2). Then, there exists a unique cubic spline interpolant $S \in \overset{\circ}{S}_4(I,\tau)$ of u which satisfies:*

$$S(t_i) = u(t_i), \quad i = 0, \cdots, n+2,$$

where $t_0 = x_0$, $t_i = \dfrac{x_{i-1} + x_i}{2}$, $i = 1, \cdots, n$, $t_{n+1} = x_{n-1}$ *and* $t_{n+2} = x_n$.

Proof: Using the Schoenberg-Whitney theorem (see [1]), it is easy to see that there exits a unique cubic spline which interpolates u_ε at the points t_i, $i=0,\cdots,n+2$.

If we put $S=\sum_{i=-3}^{n-1} c_i B_i$, then by using the boundary conditions of problem (2) we obtain $c_{-3}=S(0)=u(0)=0$ and $c_{n-1}=S(1)=u(1)=0$. Hence

$$S=\sum_{i=-2}^{n-2} c_i B_i.$$

Furthermore, since the interpolation with splines of degree d gives uniform norm errors of order $O(h^{d+1})$ for the interpolant, and of order $O(h^{d+1-r})$ for the rth derivative of the interpolant (see [1], for instance), then for any $u\in C^2(I)\cap C^0(\bar{I})$ (see [17]), we have

$$-\Delta S(t_i)=J(t_i,u)+O(1), \quad i=1,\cdots,n+1. \tag{4}$$

The cubic spline collocation method, that we present in this paper, constructs numerically a cubic spline $\tilde{S}=\sum_{i=-3}^{n-1}\tilde{c}_i B_i$ which satisfies the equation (2) at the points t_i, $i=0,\cdots,n+2$. It is easy to see that

$$\tilde{c}_{-3}=\tilde{c}_{n-1}=0,$$

and the coefficients $\tilde{c}_i$, $i=-2,\cdots,n-2$, satisfy the following nonlinear system with $n+1$ equations:

$$-\sum_{i=-2}^{n-2}\tilde{c}_i\,\Delta B_i(t_j)=J(t_j,\sum_{i=-2}^{n-2}\tilde{c}_i\,B_i(t_j)), \quad j=1,\cdots,n+1. \tag{5}$$

Relations (5) and (6) can be written in the matrix form, respectively, as follows

$$\begin{aligned}\hat{A}C&=-F-\hat{E},\\ \hat{A}\tilde{C}&=-F_{\tilde{C}},\end{aligned} \tag{6}$$

where

$$F=[J(t_1,u(t_1)),\cdots,J(t_{n+1},u(t_{n+1}))]^T,$$

$$F_{\tilde{C}}=[J(t_1,\tilde{S}(t_1)),\cdots,J(t_{n+1},\tilde{S}(t_{n+1}))]^T,$$

and $\hat{E}$ is a vector where each component is of order $O(1)$. It is well known that $\hat{A}=\frac{1}{h^2}A$, where A is a matrix independent of h, with the matrix A is invertible [2].

Then, relation (7) becomes

$$AC = -h^2 F - E,$$
$$A\tilde{C} = -h^2 F_{\tilde{C}}, \tag{7}$$

with E is a vector where each one of its components is of order $O(h^2)$.

Theorem 3.1 *Assume that the penalty parameter* λ *and the discretization parameter* h *satisfy the following relation:*

$$h^2 \lambda \| A^{-1} \|_\infty < 1. \tag{8}$$

Then there exists a unique cubic spline which approximates the exact solution u *of problem (2).*

Proof: From relation (7), we have $\tilde{C} = -h^2 A^{-1} F_{\tilde{C}}$. Let $\varphi : \mathsf{R}^{n+1} \to \mathsf{R}^{n+1}$ be a function defined by

$$\varphi(Y) = -h^2 A^{-1} F_{\tilde{Y}}. \tag{9}$$

To prove the existence of cubic spline collocation it suffices to prove that φ admits a unique fixed point. Indeed, let Y_1 and Y_2 be two vectors of R^{n+1}. Then we have

$$\| \varphi(Y_1) - \varphi(Y_2) \| \le h^2 \| A \|_\infty \| F_{Y_1} - F_{Y_2} \|_\infty . \tag{10}$$

Using relation (4) and the fact that $\sum_{j=-2}^{n-2} B_j \le 1$, we get

$$| J(t_i, S_{Y_1}(t_i)) - J(t_i, S_{Y_2}(t_i)) | \le \lambda | S_{Y_1}(t_i) - S_{Y_2}(t_i) | \le \lambda \| Y_1 - Y_2 \|_\infty .$$

Then we obtain

$$\| F_{Y_1} - F_{Y_2} \|_\infty \le \lambda \| Y_1 - Y_2 \|_\infty .$$

From relation (10), we conclude that

$$\| \varphi(Y_1) - \varphi(Y_2) \| \le \lambda h^2 \| A^{-1} \|_\infty \| Y_1 - Y_2 \|_\infty .$$

Then we have

$$\| \varphi(Y_1) - \varphi(Y_2) \| \le k \| Y_1 - Y_2 \|_\infty ,$$

with $k = \lambda h^2 \| A^{-1} \|_\infty$, by relation (8). Hence the function φ admits a unique fixed point.

In order to calculate the coefficients of the cubic spline collocation given by the nonsmooth system

$$\tilde{C} = \varphi(\tilde{C}), \tag{11}$$

we propose the generalized Newton method defined by

$$\tilde{C}^{(k+1)} = \tilde{C}^{(k)} - (I_{n+1} - V_k)^{-1}(\tilde{C}^{(k)} - \varphi(\tilde{C}^{(k)})), \tag{12}$$

where I_{n+1} is the unit matrix of order $n+1$ and V_k is the generalized Jacobian of the function $\tilde{C} \mapsto \varphi(\tilde{C})$, (see [10, 11, 12], for instance).

4. Convergence of the method

Theorem 4.1 *If we assume that the penalty parameter* λ *and the discretization parameter* h *satisfy the following relation*

$$2h^2\lambda \| A^{-1} \|_\infty < 1. \tag{13}$$

then the cubic spline $\tilde{S}$ *converges to the solution* u. *Moreover the error estimate* $\| u - \tilde{S} \|_\infty$ *is of order* $O(h^2)$.

Proof: From (7) and the matrix A is invertible [2], we have

$$C - \tilde{C} = -h^2 A^{-1}(F - F_{\tilde{C}}) - A^{-1}E.$$

Since E is of order $O(h^2)$, then there exists a constant K_1 such that $\| E \|_\infty \le k_1 h^2$. Hence we have

$$\| C - \tilde{C} \|_\infty \le h^2 \| A^{-1} \|_\infty \| F - F_{\tilde{C}} \|_\infty + K_1 \| A^{-1} \|_\infty h^2. \tag{14}$$

On the other hand we have

$$\begin{aligned} | J(t_i, u(t_i)) - J(t_i, \tilde{S}(t_i)) | &\le \lambda | u(t_i) - \tilde{S}(t_i) | \\ &\le \lambda | u(t_i) - S(t_i) | + \lambda | S(t_i) - \tilde{S}(t_i) |. \end{aligned}$$

Since S is the cubic spline interpolation of u, then there exists a constant K_2 such that (see [17]),

$$\| u - S \|_\infty \le K_2 h^2. \tag{15}$$

Using the fact that

$$| S - \tilde{S} | \le \| C - \tilde{C} \|_\infty \sum_{j=-2}^{n-2} B_j \le \| C - \tilde{C} \|_\infty, \tag{16}$$

then, we obtain

$$| F - F_{\tilde{C}} | \le \lambda \| C - \tilde{C} \|_\infty + \lambda K_2 h^2.$$

By using relation (14) and assumption (13) it is easy to see that

$$\begin{aligned} \| C-\tilde{C} \|_{\infty} &\leq \frac{h^2 \| A^{-1} \|_{\infty}}{1-\lambda h^2 \| A^{-1} \|_{\infty}} (K_2 \lambda h^2 + K_1) \\ &\leq 2 \| A^{-1} \|_{\infty} (K_2 \lambda h^2 + K_1) h^2. \end{aligned} \tag{17}$$

We have

$$\| u-\tilde{S} \|_{\infty} \leq \| u-S \|_{\infty} + \| S-\tilde{S} \|_{\infty} .$$

Then from relations (15), (16) and (17), we deduce that $\| u-\tilde{S} \|_{\infty}$ is of order $O(h^2)$. Hence the proof is complete.

Remark 4.1 Theorem 4.1 provides a relation coupling the parameter λ and the discretization parameter h, which guarantees the quadratic convergence of the cubic spline collocation $\tilde{S}$ to the solution u of the Bratu's problem (1).

5. Numerical examples

To illustrate the ability and reliability of the method for Bratu's problem, some examples are provided. The results reveal that the method is very effective and simple.

Consider the case for Bratu's equation as follows, when $\lambda = 1,\ 2\ and\ 3.51$

The maximum absolute errors in solutions of this problem are compared with methods in [18, 19, 20, 21, 22, 23, 24] for $h = 1/10$ and tabulated in Tables 1, 2 and 3. The tables show that our results are more accurate.

Table 1. Absolute errors for $\lambda = 1$.

x	Present	wavelet[24]	Spline[18]	B-Spline[19]	LGSM[20]	DTM[21]	Laplace[22]	Decmposition[23]
0.1	2.65E-13	4.22E-12	5.77E-10	2.98E-06	7.51E-07	-	1.98E-06	2.69E-03
0.2	3.40E-13	8.85E-12	2.47E-10	5.46E-06	1.02E-06	1.00E-10	3.94E-06	2.02E-03
0.3	4.16E-13	1.50E-11	4.56E-11	7.33E-06	9.05E-07	2.00E-10	5.85E-06	1.52E-04
0.4	6.93E-13	2.09E-11	9.64E-11	8.50E-06	5.24E-07	2.00E-10	7.70E-06	2.20E-03
0.5	9.71E-13	2.67E-11	1.46E-10	8.89E-06	5.07E-09	2.00E-10	9.47E-06	3.02E-03
0.6	6.93E-13	3.21E-11	9.64E-11	8.50E-06	5.14E-07	2.00E-10	1.11E-05	2.20E-03
0.7	4.16E-13	3.71E-11	4.56E-11	7.33E-06	8.95E-07	3.00E-10	1.26E-05	1.52E-04
0.8	3.40E-13	4.17E-11	2.47E-10	5.46E-06	1.01E-06	2.00E-10	1.35E-05	2.02E-03
0.9	2.65E-13	4.71E-11	5.77E-10	2.98E-06	7.42E-07	2.00E-10	1.20E-05	2.69E-03

Table 2. Absolute errors for $\lambda = 2.$

x	Present	wavelet[24]	Spline[18]	B-Spline[19]	LGSM[20]	DTM[21]	Laplace[22]	Decmposition[23]
0.1	1.46E-13	7.56E-12	9.71E-05	1.72E-05	4.03E-06	1.30E-06	2.13E-03	1.52E-02
0.2	3.32E-13	1.84E-12	1.41E-08	3.26E-05	5.70E-06	2.56E-06	4.21E-03	1.47E-02
0.3	5.30E-13	3.15E-11	1.98E-08	4.49E-05	5.22E-06	3.77E-06	6.19E-03	5.89E-03
0.4	6.12E-13	4.37E-11	2.42E-08	5.28E-05	3.07E-06	4.87E-06	8.00E-03	3.25E-03
0.5	7.46E-13	5.48E-11	2.60E-08	5.56E-05	1.46E-08	5.85E-06	9.60E-03	6.99E-03
0.6	6.12E-13	6.43E-11	2.42E-08	5.28E-05	3.05E-06	6.66E-06	1.09E-02	3.25E-03
0.7	5.30E-13	7.20E-11	1.98E-08	4.49E-05	5.19E-06	7.29E-06	1.19E-02	5.89E-03
0.8	3.32E-13	7.77E-11	1.41E-08	3.26E-05	5.68E-06	7.71E-06	1.24E-02	1.47E-02
0.9	1.46E-13	8.38E-11	9.71E-09	1.72E-05	4.01E-06	7.41E-06	1.09E-02	1.52E-02

Table 3. Absolute errors for $\lambda = 3.51$

x	Present	wavetet[24]	Spline[18]	B-Spline[19]	LGSM[20]
0.1	1.55E-10	9.34E-09	6.61E-06	3.84E-02	4.45E-05
0.2	2.47E-10	1.81E-08	5.83E-06	7.48E-02	7.12E-05
0.3	3.45E-10	2.56E-08	6.19E-06	1.06E-01	7.30E-05
0.4	4.37E-10	3.09E-08	6.89E-06	1.27E-01	4.47E-05
0.5	4.75E-10	3.30E-08	7.31E-06	1.35E-01	6.76E-07
0.6	4.37E-10	3.16E-08	6.89E-06	1.27E-01	4.56E-05
0.7	3.45E-10	2.70E-08	6.19E-06	1.06E-01	7.20E-05
0.8	2.47E-10	2.00E-08	5.83E-06	7.48E-02	7.05E-05
0.9	1.55E-10	1.17E-08	6.61E-06	3.84E-02	4.41E-05

6. Concluding remarks

In this paper, we have consider an approximation of a Bratu equation problem, presented in [19, 20]. Then we have developed a numerical method for solving each nonsmooth equation, based on a cubic collocation spline method and the generalized Newton method. We have shown the convergence of the method provided that the physical and the discretization parameters satisfy the relation (13). Moreover we have provided an error estimate of order $O(h^2)$ with respect to the norm $\|\cdot\|_\infty$. The obtained numerical results show the convergence of the approximate solutions to the exact one and confirm the error estimates provided in this paper. The analytical results are illustrated with two numerical examples. The proposed scheme is simple and computationally attractive, and shows a very high precision comparing with many other existing numerical methods.

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