Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.3, No.14, 2013


# A Note on Singular and Nonsingular Modules Relative to Torsion Theories

Mehdi Sadik Abbas[1] Mohanad Farhan Hamid[2*]

1. Department of Mathematics, College of Science, University of Mustansiriya, Baghdad, Iraq
2. Department of Mathematics, College of Science, University of Mustansiriya, Baghdad, Iraq

* E-mail of the corresponding author: mohanadfhamid@yahoo.com

**Abstract**

Let $\tau$ be a hereditary torsion theory. The purpose of this paper is to extend results about singular (resp. nonsingular) modules to $\tau$-singular (resp. $\tau$-nonsigular) modules. An *R*-module is called $\tau$-singular (resp. $\tau$-nonsigular) if all its elements (resp. none of its elements except 0) are annihilated by $\tau$-essential right ideals of *R*. We proved that, when *R* is $\tau$-nonsingular, the quotient of an *R*-module by its $\tau$-singular submodule is $\tau$-nonsingular. Goldie proved that for any submodule $N \subseteq M$, the quotient $M/N^{**}$ is nonsingular. We generalize this result to torsion theoretic setting. Also we introduce the concept of Goldie $\tau$-closure of a submodule as a generalization of Goldie closure. We proved that it is equivalent to the concept of $\tau$-essential closure in the case of $\tau$-nonsingular modules.

**Keywords:** torsion theory, torsion module, torsionfree module, $\tau$-dense submodule, (non)singular module.

## 1. Introduction

Throughout this paper we will denote by *R* an associative ring with a nonzero identity and by $\tau = (\mathcal{T},\mathcal{F})$ a hereditary torsion theory on the category Mod-*R* of right *R*-modules, where $\mathcal{T}$ (resp. $\mathcal{F}$) denotes the class of $\tau$-torsion (resp. $\tau$-torsionfree) *R*-modules. All modules considered in this paper will be right unital *R*-modules.

A submodule *N* of a module *M* is said to be $\tau$-dense in *M* (denoted $N \leq^{\tau d} M$) if *M/N* is $\tau$-torsion, and *M* is $\tau$-torsion if and only if all its elements are annihilated by $\tau$-dense right ideals of *R*. A submodule *N* of *M* is called $\tau$-essential in *M* (denoted $N \leq^{\tau e} M$) if *N* is both $\tau$-dense and essential in *M*. In this case *M* is called a $\tau$-essential extension of *N*. The intersection of any two $\tau$-dense (resp. $\tau$-essential) submodules is again a $\tau$-dense (resp. $\tau$-essential) submodule. Any submodule that contains a $\tau$-dense (resp. $\tau$-essential) submodule is itself $\tau$-dense (resp. $\tau$-essential). For any torsion theory $\tau$ there corresponds a radical *t* such that for every module *M*, $t(M)$ is the largest $\tau$-torsion submodule of *M*. The module *M* is $\tau$-torsion (resp. $\tau$-torsionfree) if and only if $t(M) = M$ (resp. $t(M) = 0$). Every module *M* admits a $\tau$-injective envelope, i.e. a $\tau$-injective module containing *M* as a $\tau$-essential submodule. For preliminaries about torsion theories, we refer to Bland (1998).

For any module *M* there is defined a submodule Z(*M*) which consists of singular elements in *M*, i.e. elements annihilated by essential right ideals. The module *M* is singular (resp. nonsingular) according to whether Z(*M*) = *M* (resp. Z(*M*) = 0), see Goldie (1964).

Charalambides (2006) introduced the concept of $\tau$-essentially closed submodules. A submodule *N* of a module *M* is called $\tau$-essentially closed in *M* (denoted $N \leq^{\tau c} M$ if *N* has no proper $\tau$-essential extensions in *M*.

In this paper we introduce the concept of (non)singularity in a torsion theoretic setting. We say that an element *m* in a module *M* is $\tau$-singular if its right annihilator is a $\tau$-essential right ideal in *R*. The module *M* is $\tau$-singular (resp. $\tau$-nonsingular) if all its elemenets are $\tau$-singular (resp. if the only $\tau$-singular element is 0). The notions of $\tau$-singularity and $\tau$-nosingularity reduce to singularity and nonsigularity respectively if the torsion theory $\tau$ is chosen so that every *R*-module is $\tau$-torsion. Generalizing the idea of Goldie closure, we introduce the concept of Goldie $\tau$-closure of a submodule and compare it to that of $\tau$-essentially closed submodules of Charalambides (2006). We will prove that they are equivalent in the case that the module is $\tau$-nonsingular. Finally, by ann($x$), where $x$ is an element in some given module *M*, we mean the right annihilator ideal of $x$. If we want to emphasize the side (if our module is two-sided), we write $\text{ann}_r(x)$ (resp. $\text{ann}_l(x)$) for the right (resp. left) annihilator of the element $x$. By $(M{:}x)$ we mean the set $\{r \in R \mid xr \in M\}$. And by $N \leq^{e} M$ we mean that *N* is an essential submodule of *M*.

## 2. τ-Singular and τ-Nonsingular Modules

**Definition.** Let $M$ be a right module over a ring $R$. An element $m$ in $M$ is said to be a τ-singular element of $M$ if the right ideal ann($m$) is τ-essential in $R_R$. The set of all τ-singular elements of $M$ is denoted by $Z_\tau(M)$. We say that $M$ is a τ-singular (resp. τ-nonsingular) module if $Z_\tau(M) = M$ (resp. $Z_\tau(M) = 0$). In particular, we say that $R$ is a right τ-nonsingular ring if $Z_\tau(R) = 0$.

Note that elements of $Z_\tau(M)$ are all singular and $Z_\tau(M)$ is τ-torsion. So we have that

$$Z_\tau(M) = Z(M) \cap t(M) = t(Z(M)) = Z(t(M)) \qquad \ldots\ldots. (1)$$

The following proposition characterizes τ-singular modules.

**Proposition 1.** A module $M$ is τ-singular if and only if it is both singular and τ-torsion.

*Proof.* Let M be τ-singular, then $Z_\tau(M) = M$. But $Z(M) \supseteq Z_\tau(M) = M$. So $M$ must be singular. Since $Z_\tau(M)$ is τ-torsion, so is $M$. To prove the converse apply the equalities $Z(M) = M$ and $t(M) = M$ to (1).

In the following proposition, we give some properties of $Z_\tau(M)$.

**Proposition 2.**

($i$) $Z_\tau(M)$ is a submodule of $M$, called the τ-singular submodule of $M$.

($ii$) $Z_\tau(M).\mathrm{soc}^\tau(R) = 0$, where $\mathrm{soc}^\tau(R) = \cap\{I \mid I \leq^{\tau e} R\}$.

($iii$) If $f : M \to N$ is an $R$-homomorphism, then $f(Z_\tau(M)) \subseteq Z_\tau(N)$.

($iv$) If $M$ is a submodule of $N$, then $Z_\tau(M) = M \cap Z_\tau(N)$.

($v$) If $M_i$ are modules ($i \in I$) then $Z_\tau(\oplus M_i) = \oplus Z_\tau(M_i)$.

*Proof.* ($i$) Clear. ($ii$) For any $m \in Z_\tau(M)$ we have $\mathrm{ann}(m) \leq^{\tau e} R$, so $\mathrm{soc}^\tau(R) \subseteq \mathrm{ann}(m)$. This shows that $m.\mathrm{soc}^\tau(R) = 0$. ($iii$) This follows from the fact that $\mathrm{ann}(m) \subseteq \mathrm{ann}(f(m))$ for any $m \in M$, and ($iv$) follows directly from the definition. ($v$) Let $(m_i)_I \in Z_\tau(\oplus M_i)$ then $K := \mathrm{ann}((m_i)_I) \leq^{\tau e} R$. But $K \subseteq \mathrm{ann}(m_i)$ for each $i$. So we must have $\mathrm{ann}(m_i) \leq^{\tau e} R$ for each $i$. This means that $(m_i)_I \in Z_\tau(M_i)$. Conversely, let $(m_i)_I \in \oplus Z_\tau(M_i)$. Then for each $i$ we have $\mathrm{ann}(m_i) \leq^{\tau e} R$. Since $m_i \neq 0$ for only a finite subset $J$ of $I$, then $\mathrm{ann}((m_i)_I) = \cap_{i \in J}\mathrm{ann}(m_i)$ and is indeed τ-essential in $R$ since it is the finite intersection of τ-essential right ideals in $R$.

**Corollary.** (1) $Z_\tau(R_R)$ is a (two-sided) ideal of $R$, called the right τ-singular ideal of $R_R$. (2) If $R \neq 0$, then $Z_\tau(R_R) \neq R$.

*Proof.* By part ($i$) of the above proposition, we need only show that $m \in Z_\tau(R)$ and $s \in R$ imply that $sm \in Z_\tau(R)$. This is clear from the fact that $\mathrm{ann}_r(sm) \supseteq \mathrm{ann}_r(m)$. (2) $\mathrm{ann}_r(1) = 0$ cannot be τ-essential in $R$ unless $R = 0$.

**Proposition 3.** The quotient module $Z(M)/Z_\tau(M)$ is τ-torsionfree.

*Proof.* We must show that $t(Z(M)/Z_\tau(M)) = 0$. But $t$ is a radical, hence $t(Z(M)/Z_\tau(M)) = t(Z(M))/Z_\tau(M)$ which by (1), is equal to $Z_\tau(M)/Z_\tau(M) = 0$.

**Examples.**

(1) Any nonsingular module is τ-nonsingular.

(2) Let $R$ be a ring in which every τ-dense right ideal is essential. Then, for any $R$-module $M$, we have $Z_\tau(M) = \{m \in M \mid \mathrm{ann}(m) \leq^{\tau e} R\}$ is just the τ-torsion submodule of $M$. In particular, $M$ is τ-singular iff it is τ-torsion, and it is τ-nonsingular iff it is τ-torsionfree. Conversely, if every τ-torsion module is τ-singular then every τ-dense right ideal of $R$ is essential. To see this, let $I \leq^{\tau d} R$. So $R/I$ is τ-torsion, and hence a τ-singular right $R$-module whose elements are annihilated by τ-essentail right ideals. In particular, $\mathrm{ann}(1+I) = I$ is (τ-)essential.

(3) Let $M \subseteq N$ be $R$-modules. If $N$ is τ-nonsingular, so is $M$, and the converse is true if $M \leq^{e} N$. The first part follows from Proposition 2. For the converse, suppose that $Z_\tau(M) = 0$ and $M \leq^{e} N$. Then by Proposition 2, we have $Z_\tau(N) \cap M = Z_\tau(M) = 0$ and so $Z_\tau(N) = 0$. In particular, we see that $M$ is τ-nonsingular iff its (τ-)injective envelope is τ-nonsingular.

(4) An $R$-module $M$ is τ-singular iff there exist $R$-modules $A \leq^{\tau e} B$ such that $M \cong B/A$. This follows from the corresponding result on singular modules in p.247 of Lam (1998) and Proposition 3 above.

(5) Call a ring $R$ right $\tau$-split if every $\tau$-dense right ideal is a direct summand. So $R$ is right $\tau$-split iff every $R$-module is $\tau$-nonsingular. (In particular, a right $\tau$-split ring is right $\tau$-nonsingular.) To see this assume that $R$ is $\tau$-split and let $m \in Z_\tau(M)$, hence $\text{ann}(m) \leq^{\tau e} R$ implies that $\text{ann}(m) = R$, so $m = 0$. Conversely, suppose every $R$-module $M$ is $\tau$-nonsingular. For every $\tau$-dense right ideal $I$ of $R$, let $J$ be a complement, so that $I \oplus J \leq^{\tau e} R$. Then $R/(I \oplus J)$ is a $\tau$-singular right module by example (4), and so $I \oplus J = R$. Thus, every $\tau$-dense right ideal is a direct summand.

Examples of $\tau$-nonsingular rings are given in the following:

**Remark.** If $R$ is a ring such that no maximal right ideal is $\tau$-essential, then $R$ is $\tau$-nonsingular.

*Proof.* Suppose $Z_\tau(R) \neq 0$. Then there exists a non-zero element $r \in R$ such that $\text{ann}(r) \leq^{\tau e} R$. Now $1 \notin \text{ann}(r)$ hence $\text{ann}(r)$ is contained in some maximal right ideal $M$ which again must be $\tau$-essential in $R$. A contradiction with the assumption.

**Proposition 4.** Let $R$ be a $\tau$-nonsingular ring and $P$ a projective $R$-module containing a finitely generated $\tau$-essential submodule. Then $P$ must be finitely generated.

*Proof.* By the Dual Basis Lemma, there is a family $\{x_i\}$ of elements of $P$ and a family $\{f_i\} \subseteq \text{Hom}(P,R)$ such that for $x \in P$, $x = \sum x_i f_i(x)$, where $f_i(x) = 0$ for all but a finite number of $i$'s. Clearly, it is sufficient to show that $f_i$ is the zero map for all but finitely many $i$'s. Let $B$ be a finitely generated $\tau$-essential submodule of $P$ with generators $b_1, \ldots, b_n$, then the set $A = \{i \mid f_i(b_j) \neq 0 \text{ for some } 1 \leq j \leq n\}$ is a finite subset of the indices and $f_i(B) = 0$ if $i \notin A$. If $x \in P$, then $I = \{r \in R \mid xr \in B\}$ is a $\tau$-essential right ideal of $R$ and for $i \notin A$, $0 \neq f_i(xr) = f_i(x)r$ for every $r \in I$, so $f_i(x) \in Z_\tau(R) = 0$ and the result follows.

The next result gives examples of rings that fail to be $\tau$-nonsingular.

**Proposition 5.** Let $x$ be a central nilpotent element in a ring $R$, such that $\text{ann}(x) \leq^{\tau d} R$. Then $x \in Z_\tau(R_R)$.

*Proof.* To show that $\text{ann}(x) \leq^{e} R$, let $y$ be any nonzero element in $R$. There exists a smallest $n \geq 0$ such that $x^{n+1}y = 0$. Then $x^n y \in \text{ann}_r(x)\backslash\{0\}$. Since $x^n y = yx^n$, we have shown that $\text{ann}(x) \leq^{e} R$.

The following result shows that $M/Z_\tau(M)$ is $\tau$-nonsingular provided that $R$ is $\tau$-nonsingular.

**Theorem 6.** Let $R$ be a right $\tau$-nonsingular ring and $M$ be any $R$-module. Then $Z_\tau(M/Z_\tau(M)) = 0$.

*Proof.* We will show that $Z_\tau(M/Z_\tau(M)) \subseteq Z(M)/Z_\tau(M)$. Hence the result follows since a $\tau$-torsion submodule of a $\tau$-torsionfree module must be 0. So let $m \in M$ be such that $m + Z_\tau(M) \in Z_\tau(M/Z_\tau(M))$. Then $mI \subseteq Z_\tau(M)$ for some right ideal $I \leq^{\tau e} R$. To show that $m \in Z(M)$, we must show that $\text{ann}(m) \leq^{e} R$. Let $J \neq 0$ be any right ideal in $R$. Fixing a nonzero element $x \in I \cap J$, we have $mx \in mI \subseteq Z_\tau(M)$, so $mxK = 0$ for some right ideal $K \leq^{\tau e} R$. But $xK \neq 0$, for otherwise $x \in Z_\tau(R)\backslash\{0\}$. Therefore $xy \in \text{ann}(m) \cap J$. This shows that $\text{ann}(m) \leq^{e} R$, as desired.

If $R$ is not $\tau$-nonsingular then $M/Z_\tau(M)$ may not be $\tau$-nonsingular. For an example where $\tau$ is the improper torsion theory, see p.254 of Lam (1998). Recall that for any submodule of an $R$-module $M$, there is defined a module $N^*$ as the (unique) submodule of $M$ containing $N$ such that $N^*/N = Z(M/N)$. Generalizing this, we give the following definition:

**Definition.** Let $N$ be any submodule of an $R$-module $M$, we define $N'$ to be the submodule of $M$ containing $N$ such that $N'/N = Z_\tau(M/N)$. This process can be repeated, so we can define $N''$, $N'''$, and so on.

Notice that $N'/N$, $N''/N$, … etc. are always $\tau$-torsion. It is clear that $N' \subseteq N^*$ and $0'$ is just $Z_\tau(M)$. Moreover, we have

$$N' = \{y \in M \mid (N:y) \leq^{\tau e} R\} \qquad \text{……..(2)}$$

$$N'' = \{y \in M \mid (N':y) \leq^{\tau e} R\}, \text{ etc.} \qquad \text{……(3)}$$

Using (2), we see immediately that:

**Lemma 7.** If $L \subseteq N \subseteq M$, then $L' \subseteq N'$. In particular, $Z_\tau(M) = 0' \subseteq N'$.

Goldie (1964) proved that for any submodule $N \subseteq M$, the quotient $M/N^{**}$ is nonsingular. Generalizing this, we will prove that $M/N''$ is always $\tau$-nonsingular. But before this, we need a generalization of the idea of essential extensions, also due to Goldie. We say that two submodules $S, T \subseteq M$ are related (written $S \sim T$) if, for any submodule $X \subseteq M$, we have $X \cap S \neq 0$ iff $X \cap T \neq 0$. Clearly, "~" is an equivalence relation on the submodules of $M$. If $S \subseteq T$ then $S \sim T$

simply gives $S \leq^{e} T$.

Some basic properties of the equivalence relation "~" are given here.

**Proposition 8.** Let $L$ and $N$ be submodules of a module $M$. Then:

(i) $N + 0' \sim N'$.

(ii) $N' \sim N''$.

(iii) If $L \sim N$ and $L \cap N \leq^{\tau d} L$ then $L \subseteq N'$.

*Proof.* (*i*) We want to check that $N + 0' \leq^{e} N'$. Let $X$ be a submodule of $N'$ such that $X \cap (N+0') = 0$. For any $x \in X$ there is a right ideal $I \leq^{\tau e} R$ such that $xI \subseteq N$. Then $xI \subseteq X \cap N = 0$ implies that $x \in X \cap 0' = 0$, and hence $X = 0$.

(*ii*) Replacing $N$ by $N'$ in (*i*), we get $N'' \sim (N'+0') = N'$ (i.e. $N' \leq^{e} N''$).

(*iii*) Since $L \sim N$ then by p. 255 of Lam (1998), we have $L \subseteq N^{*}$, i.e. $(N{:}l) \leq^{e} R$ for any $l \in L$. But $(N{:}l) = \{r \in R \mid lr \in N\} = \{r \in R \mid lr \in L \cap N\}$ is a $\tau$-essential right ideal in $R$ since $L \cap N \leq^{\tau d} L$. This shows that for any $l \in L$, we have $(N{:}l) \leq^{\tau e} R$, hence $l \in N'$.

We are now ready to prove the generalization of Goldie's Theorem:

**Theorem 9.** For any submodule $N \subseteq M$, we have $N''' = N''$. In other words $M/N''$ is $\tau$-nonsingular.

*Proof.* Let $N \subseteq N'$. Replacing $N$ by $N'$ in part (*ii*) of Proposition 8, we get $N''' \sim N'' \sim N'$. Applying part (*iii*) with $N$ replaced by $N'$, and $L = N'''$ and noticing that $N' \cap N''' = N'$, hence $N'''/(N' \cap N''') = N'''/N' \cong (N'''/N)/(N'/N)$ is $\tau$-torsion, we get $N''' \subseteq N''$, and hence $N''' = N''$.

**Corollary.** For any submodule $N \subseteq M$, the module $N''$ has no $\tau$-essential extension in $M$.

*Proof.* Consider any submodule $Y$ such that $N'' \leq^{\tau e} Y \leq M$. Then by part (*iii*) of Proposition 8 we get $Y \sim N''$ and $Y \cap N'' = N'' \leq^{\tau d} Y$ imply that $Y \subseteq N''' = N''$, so $Y = N''$.

**Definition.** For any submodule $N \subseteq M$, we write $\mathrm{cl}_{\tau}(N) := N''$, and call this the (Goldie) $\tau$-closure of $N$ in $M$.

Note that in the special case when $R$ is a right $\tau$-nonsingular ring, $\mathrm{cl}_{\tau}(N) = N'$ by Theorem 6.

We now finish with some remarks about $\tau$-nonsingular modules. If $N \subseteq M$ where $M$ is any module, then $N$ need not be $\tau$-essential in $N'$. For instance, (0) is not essential in $0'$, unless $M$ is $\tau$-nonsingular. In case $M$ is $\tau$-nonsingular, the prime operation behaves in a much nicer way. We summarize the relevant facts in the proposition below. Note in particular that, in a $\tau$-nonsingular module, the notions of Goldie $\tau$-closure and $\tau$-essential closure coincide.

**Proposition 10.** Let $M$ be a $\tau$-nonsingular right $R$-module, and let $N \subseteq M$. Then:

(a) $N \leq^{\tau e} N'$.

(b) $N'$ is the largest submodule of $M$ with the properties that $N' \sim N$ and $N' \cap N \leq^{\tau d} N'$.

(c) $N'$ is the smallest $\tau$-essentially closed submodule of $M$ containing $N$. (In particular, $N'$ is the $\tau$-essential closure of $N$ in $M$.)

(d) $N'' = N'$.

(e) $N = N'$ iff $N$ is $\tau$-essentially closed in $M$.

(f) If $N_i$ is $\tau$-essentially closed in $M$ ($i \in I$) then $\cap N_i$ is $\tau$-essentially closed in $M$.

*Proof.* Since $0' = 0$, part (*i*) of Proposition 8 gives $N \sim N'$, proving (a). If $L \sim N$ and $L \cap N \leq^{\tau d} L$, part (*iii*) of Proposition 8 gives $L \subseteq N'$, proving (b). From Proposition 8 part (*ii*), we have $N'' \sim N' \sim N$ so (b) shows that $N'' = N'$, proving (d). To prove (e), first assume $N = N'$. Then $N = N''$ is $\tau$-essentially closed in $M$ by Corollary of Theorem 9. Conversely, if $N$ is $\tau$-essentially closed in $M$, then (a) implies that $N' = N$. To prove (c), note that $N' = N''$ is $\tau$-essentially closed in $M$. Moreover, if $N \subseteq X \leq^{\tau c} M$, then $N' = X' = X$ by (e). For (f), let $N = \cap N_i$. We have an injection $M/N \to \oplus M/N_i$. Since $N_i' = N_i$ by (e), each $M/N_i$ is $\tau$-nonsingular. By part (*v*) of Proposition 2, we have $\oplus M/N_i$ is $\tau$-nonsingular, and by Proposition 2 part (*iv*), so is $M/N$. This means that $N = N'$, so by (e) again, $N \leq^{\tau c} M$.

**5. Acknowledgement**

The authors would like to express their gratitude to Professor Edgar Enochs who has read the paper and provided his valuable remarks.

**References**

Bland, P.E. (1998), “Topics in Torsion Theory”, Wiley-VCH, Berlin.

Charalambides, S. (2006), “Topics in Torsion Theory”, PhD Thesis, University of Otago, Dunedin, New Zealand.

Goldie, A.W. (1964), “Torsion-Free Modules and Rings”, *Journal of Algebra* **1**, 268-287.

Lam, T.Y. (1999), “Lectures on Modules and Rings”, Springer-Verlag, New York.