Seifert- Van Kampen Theorem

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Abstract

In this paper, the Seifert – Van Kampen Theorem deals with the situation where a path-connected space may be written as the union of two open path-connected intersection. The fundamental group of the space is isomorphic to the co-limit of the diagram of the fundamental groups of the three subspaces.

We also considered the fact that the fundamental groups are all injective, the fundamental group of the space is a classical free product with amalgamation and also a free product with the amalgamation in the category of rings.

Keywords: Fundamental groups, Homotopy, Homomorphism, Seifert-Van Kampen, Simply Connected, Knot Theory.

1. Introduction

We have actually determined the structure of the fundamental group of only a very few spaces (e.g., contractible space, the circle). To be able to apply the fundamental group to a wider variety of problems, we must learn methods for determining its structure for more spaces.

The Van Kampen theorem gives a method for computiong the fundamental groups of spaces that can be decomposed into simpler spaces whose fundamental groups are already known.

Schwanzl et al(2002) assume that to determined the fundamental group of an arcwise-connected space X, which is the union of two subspaces U and V, each of which is arcwise connected, and whose fundamental group is known, we choose a point $x_0 \in U \cap V$, the Van Kampen theorem asserts that, if U andn V are both open sets, and it is assumed that their intersection $U \cap V$ is arcwise connected, then $\pi(X,x_0)$ is completely determined by the following diagram of groups and homomorphism

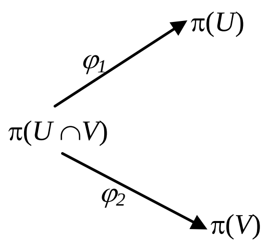


Figure 1

Here φ_1 and φ_2 are induced by inclusion maps. The way in which $\pi(X,x_0)$ is determined by this diagram can be roughly described as follows. The above diagram can be completed by forming the following commutative diagram.

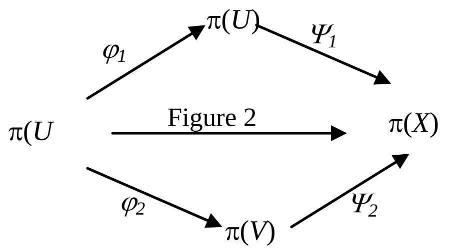


Figure 2

Here all arrows denote homomorphisms induced by inclusion maps, and the base point x_0 is systematically

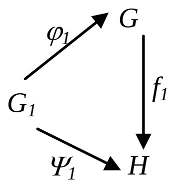
omitted. The Van Kampen theorem asserts that $\pi(X)$ freest possible group we can use to complete diagram (Figure 1) to a commutative diagram like Figure 2.
We shall state and prove a more general version of the therem, in that we allow X to be the union of any number of arcwise connected open subsets rather that just two.
The statement of the Van Kampen theorem will be in terms of free products of groups, so before starting the theorem we shall make an algebraic digression to describe the construction of free products in some details.

2. Free Products of Groups

The free product of a collection of groups is the exact analog for arbitrary groups of the weak product for Abelian groups.

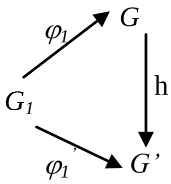
Definition 2.1

Let $\{G_1 : i \in I\}$ be a collection of groups wan assume that there is given for each index i a homomorphism φ_1 of G_1 into a fixed group G. We say that G is the free product of the groups G_1 (with respect to the homomorphisms φ_i) if and only if the following condition holds:
For any group H and any homomorphisms
$\Psi_1 : G_i \rightarrow H, i \in I,$
there exists a unique homomorphism $f : G \rightarrow H$ such that for any $i \in I$, the following diagram is commutative.



The following uniqueness proposition about free products is as follows:

Proposition 2.2
Assume that G and G' are free products of a collection $\{G_i : i \in I\}$ of groups (with respect to homomorphisms $\varphi_1 : G_1 \rightarrow G$ and $\varphi_1' : G_1 \rightarrow G'$ respectively). Then, there exists a unique isomorphism $h: G \rightarrow G'$ such that the following diagram is commutative for any $i \in I$:



since we have defined free products of groups and also stated its uniqueness, it remains to prove that they always exist.

Theorem 2.3

Given and collection $\{G_i : i \in I\}$ of groups, the free product exists.

Proof
We define a word in the G_1's to be a finite sequence $(x_1, x_2, \ldots, x_n)$ where each x_k belongs to the groups G_1, any two successive terms in the sequence sequence belong to different groups, and no term is the identity element of any G_1. The integer n is the length of the word. Brown et al(2011) also include the empty word, i.e. the unique word of length 0. Let W denote the set of all such words.
For each index I, we now define left operations of the group G_i on the set W. Let $g \in G_i$ and
$(x_1, x_2, \ldots, x_n) \in W$; we must define $g(x_1, x_2, \ldots, x_n)$.

Case 1: $x_1 \notin G_1$.Then, if $g \neq 1$.
$g(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n)$.

We shall also define the action of g on the empty word by a similar formula, i.e.,
g( ) = (g). If $g \neq 1$, then

$g(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n)$.

Case 2: $x \in G_1$. Then,

$$g(x_1, \ldots, x_n) = \begin{cases} (gx_1, x_2, \ldots, x_n) & \text{if } gx_1 \neq 1 \\ (x_2, \ldots, x_n) & \text{if } gx_1 = 1 \end{cases}$$

(When $gx_1 = 1$ and $n = 1$, it is understood, of course, that $g(x_1)$ is the empty word)
We must now verify that the requirements for the operations of G_1 on W are actually satisfied, i.e., for any word w,

$1w = w$,

$(gg')w = g(g'w)$

It is clear that each element g of G_i may be considered as a permutation of the set W, and G_1 may be considere as a subgroup of all permutations of W. Let G denote the subgroup of the group of all permutations of W which is generated by the unon of the G_1's. Then, G contains each G_1 as a subgroup; we let,

$\varphi_1 : G_1 \rightarrow G$

denote the inclusion map.
Any element of G may be expressed as a fuinite product of elements from the various G_1's. If two consecutive factors in this product come from the same G_1, it is clear that they may be replaced by a single factor. Thus, any element $g \neq 1$ of G may be expressed as a finite product of elements from the G_1's in reduced form, i.e., so no two consecutive factors belong to the same group, and so no factor is the identity element. We now assert that the expression of any element $g \neq 1$ of g in reduced form is unique. If

$g = g_1 g_2 \ldots g_m = h_1 h_2 \ldots h_n$

with both products in the reduced form, then $m=n$ and $g_i = h_i$ for $1 \leq I \leq m$.
We consider the effect of the permutation $g_1 g_2 \ldots g_m$ and $h_1 h_2 \ldots h_m$ on the empty word; the result are the words $(g_1, g_2, \ldots, g_m)$ and $(h_1, h_2, \ldots, h_m)$, respectively. Because these two words must be equal, the conclusion follows.
We can now verify that G is actually the free product of the G_1's with respect to the φ_1's. Let H be any group and let $\Psi_1 : G_1 \rightarrow H$, $I \in I$, be any collection of homomorphisms. We define a function $f : G \rightarrow H$ as follows. We express any given $g \neq 1$ in reduced form,

$g = g_1 g_2 \ldots g_m$, $g_k \in G_{1k}$, $1 \leq I \leq m$.

and then set

$f(g) = (\Psi_{i1} g_1)(\Psi_{i2} g_2) \ldots (\Psi_{im} g_m)$

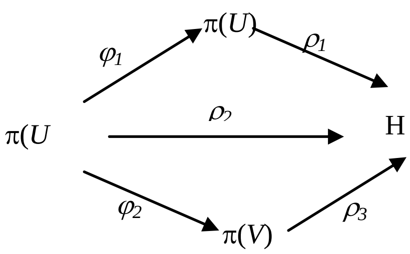
We also set $f(1) = 1$, of course. It is clear that f is a homomorphism, and that f makes the required disgrams commutative. It is also clear that f is the only homomorphism that makes these diagrams commutative.

3. Statement of The Seifert-Van Kampen's Theorem

Brown et al(1987) give a precise statement of the theorem. We assume that U and V are arcs/wise-connected open subsets of X such that $X = U \cup V$ and $U \cap V$ is non-empty any arcwise-connected. We choose a basepoint $x_0 \in U \cap V$ for all fundamental groups under consideration.

Theorem 3.1

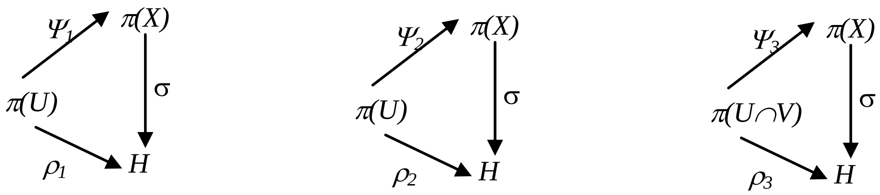
Let H be any group, and ρ_1, ρ_2 and ρ_3 any three homomorphisms such that the following diagram is commutative:



Then, there exists a unique homomorphism

$\sigma : \pi(X) \rightarrow H$

Such that the following three diagrams are commutative.



(Here the homomorphisms φ_1 and Ψ_i, I = 1, 2, 3 are induced by inclusion maps)
The statement of the more general version of the theorem consists in allowing a covering of the space X by any number of open sets instead of just by two open sets as in

Theorem 3.2

The open set must all be arcwise-connected and the intersection of any finite number of them must be arcwise-connected and contain the basepoint. We must assume the following hypotheses.

(a) X is an arcwise-connected topological space and $x_0 \in X$.
(b) $\{U_\lambda: \lambda \in \Lambda\}$ is a covering of by arcwise-connected open sets such that for all $\lambda \in \Lambda, x_0 \in U_\lambda$.
(c) For any two indices $\lambda_1, \lambda_2 \in \Lambda$ there exists an index $\lambda \in \Lambda$ such that $U_{\lambda_1} \cap U_{\lambda_2} = U_\lambda$
(d) (we express this by saying that the family of sets $\{U_\lambda\}$is "closed under finite intersectios").

Ronald et al(2006) consider the fundamental groups of these various sets with base point x_0. For brevity, we omit the basepoint from the notation.

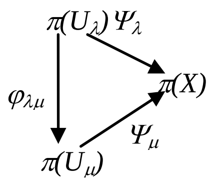
If $U_\lambda \subset U_\mu$, then the notation

$$U_{\lambda\mu}: \pi(U_\lambda) \to \pi(U_\mu)$$

denotes the homomorphism induced by the inclusion map. Similarly, for any index λ,

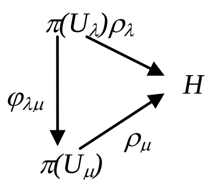
$$U_\lambda: \pi(U_\lambda) \to \pi(X)$$

is induced by the inclusion map $U_\lambda \to X$. We note that if $U_\lambda \subset U_\mu$, the following diagram is commutative.

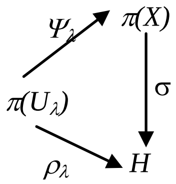


Theorem 3.3

Under the above hypothesis the group $\pi(X)$ satisfies the following universal mapping condition:
Let H be any group and let $\rho_\lambda: \pi(U_\lambda) \to H$ be any collection of homomorphisms defined fo all $\lambda \in \Lambda$ such that if $U_\lambda \subset U_\mu$ the following diagram is commutative.



Then, there exists a unique homomorphism $\sigma : \pi(X) \to H$ such that for any $\lambda \in \Lambda$ the following diagram is commutative.



Moreover, this universal mapping condition characterizes $\pi(X)$ up to a unique isomorphism.

4. Proof of The Seifert-Van Kampen's Theorem

Lemma 4.1

The group $\pi(X)$ is generated by the unuion of the images $\Psi_\lambda[\pi(U_\lambda)], \lambda \in \Lambda.$

Proof

Let $\alpha \in \pi(X)$, choose a pth $f : I \rightarrow X$ representing α. We choose an interger n so large that $\frac{1}{n}$ is less than the Lebesgue number of the open covering $\{f^{-1}(U_\lambda), \lambda \in \Lambda.$ of the copact metric space I. Subdividing the interval I into the closed subintervals $J_1 = \left[\frac{1}{n}, \frac{(i+1)}{n}\right], 0 \leq i \leq n-1.$ For each suinterval J_i, we choose an index $\lambda_i \in \Lambda$ such that $f(J_i) \subset U_\lambda$. We choose a path g_i in $U_{\lambda_{i-1}} \cap U_\lambda$, joining the point x_0 to the point $f(i/n),\ 1 \leq i \leq n-1. Let\ f_1 : I \rightarrow X$ denote the path represented by the composite function

$$I \xrightarrow{h_1} J_1 \xrightarrow{f|J_1} X$$

Where h_i is the unique orientation-preserving linear homeomorphism. Then $f_0 . g_1^{-1}, g_1 . f_1 . g_2^{-1}, g_2 . f_2 . g_3^{-1}, \dots\ g_{n-2} . f_{n-2} . . g_{n-1}^{-1}, g_{n-1} . f_{n-1}$ are closed paths, each contained in a single open set U_λ and their product in the order given is equivalent to f.

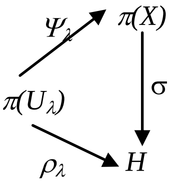
Hence, we can write

$$\alpha = \alpha_0 . \alpha_1 . \alpha_2 \dots . \alpha_{n-1}$$

Where $\alpha_1 \in \Psi_{\lambda,} [\pi(U_\lambda)], 0 \leq I \leq n-1.$

Proof of the Theorem

Let H be any group and let ρ_λ:$\pi(U_{\lambda,}) \rightarrow H$, $\lambda \in \Lambda$ be a set of homomorphisms satisfying the hypotheses of the theorem. We must demonstrate the existence of a unique homomorphism $\sigma : \pi(X) \rightarrow H$ such that the following diagram is commutative for any $\lambda \in \Lambda$



From the Lemma just proved, it is clear that such a homomorphism σ, if it exists, must be unique, and must be defined according to the following rule.

Let $\alpha \in \pi(X)$. Then, by **Lemma 3.3.**, we have

$$\alpha = \Psi_{\lambda_1}(\alpha_1).\ \Psi_{\lambda_2}(\alpha_2)\ldots\ \Psi_{\lambda_n}(\alpha_n) \qquad (1)$$

Where $\alpha_1 \in \pi(U_\lambda),\ i = 1, 2, \ldots, n$. Hence, if the homomorphism σ exists, we must have

$$\sigma(\alpha) = \rho_{\lambda_1}(\alpha_1).\ \rho_{\lambda_2}(\alpha_2)\ldots\ \rho_{\lambda_n}(\alpha_n) \qquad (2)$$

The strategy involved is to take equation (2) as a definition of σ. To justify this definition, we must show that it is indepenedent of the choice of the representation of α in the form (1). Clearly, if it is indepenedent of the form of the representation of α, then it is a homomorphism, and the desired commutativity relations must hold.

To prove that σ is independent of the representation of α in the form of (1), it suffices to prove the following lemma.

Lemma 4.2

Let $\beta_1 \in \pi(U_\lambda),\ i = 1, 2, \ldots,$ q be such that

$$\Psi_{\lambda_1}(\beta_1).\ \Psi_{\lambda_2}(\beta_2)\ldots\ \Psi_{\lambda_q}(\alpha_q) = 1$$

Then the product

$$\rho_{\lambda_1}(\beta_1).\ \rho_{\lambda_2}(\beta_2)\ldots\ \rho_{\lambda_q}(\beta_q)$$

Proof

We choose closed paths

$$f_1 : \left[\frac{i-1}{q}, \frac{i}{q}\right] \to U_{\lambda_1}$$

representing β_1 for I = 1, 2, …, q. Then the product

$$\prod_{1=1}^{q} \Psi_{\lambda_1}(\beta_1)$$

is clearly represented by the closed path $f, :[0, 1] \to$ X defined b

$$f_1 \left| \left[\frac{i-1}{q}, \frac{i}{q}\right] = f_1; i = 1, 2, \ldots, q.\right.$$

By hypothesis, f is equivalent to the constant path. Hence, there exists a continuous map

$$F : I \text{ x } I \to X$$

such that, for any $s, t \in I$,

$$F(s,0) = f(s)$$
$$F(s, 1) = F(0,t) = F(1,t) = x_0$$

Let ε denote the Lebesque number of the open covering $\{F^{-1}(U_\lambda) : \lambda \in \Lambda\}$ of the compact metric space *I x I*. (We give *I x I* the metric it has as a subset of the Euclidean plane). We now subdivide the square I xI into smaller rectangles of diameter < ε as follows. We choose numbers.

$$s_0 = 0, s_1, s_2, \ldots, s_m = 1$$
$$t_0 = 0, t_1, t_2, \ldots, t_n = 1$$

the length of the diagonal of each rectangle is less than ε. Clearly, a subdivision is possible. We shall introduce notation for the various vertices, edges and rectangles of this subdivision as follows:

vertices:

$$v_q = (s_1, t), 0 \le i \le m, 0 \le j \le n$$

subdintervals of $I = [0, 1]$

$$J_1 = [s_{t-1}, s_1], I \le i \le m,$$
$$K_1 = [t_{t-1}, t_1], I \le i \le n,$$

Rectangles:

$$R_q = J_1 \text{ x } K_1, 1 \le i \le m, 1 \le j \le n,$$

Horizontal edges;

$a_q = J_i \times \{t_j\}, 1 \le i \le m, 0 \le j \le n,$

Vertical edges:

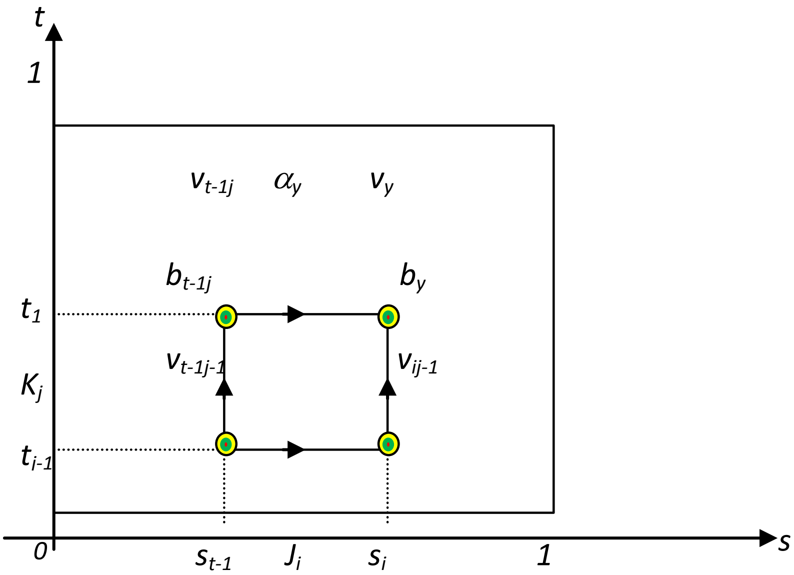
$b_q = \{s_t\} \times K_t, 0 \le i \le m, 1 \le j \le n.$

we indicate how a typical rectangle of the subdivision and its vertices andedges are labeled. We also need the following notation for certain paths:

$A_q : J_t \to X, A_q(s) = F(s,t), s \in J$

$B_q : K_t \to X, B_q(t) = F(s,t), t \in K$

For each rectangle R_q, we choose an open set $U_\lambda(I, j)$ such that $F(R_{i,j}) \subset U_{\lambda(I,j)}$



Each vertex v_q is a vertex of 1, 2, or 4 of the rectangles R_{kl}. Let $U_{\mu(I,j)}$ denote the intersection of the corresponding 1, 2, or 4 open sets $U_{\lambda(k,j)}$. Then, is an open set of the given covering and

$F(v_{ij}) \in U_{\mu(t,j)}$

we choose a path

$g_n : I \to U_{\mu(t,1)}$

with inital point x_0 and terminal point $F(v_{ij})$; if $F(v_{ij}) = x_0$, we require that g_y be the contant path.

We shall now interpolate a sublemma.

Sublemma 4.3

Let U_λ and U_μ be two sets of the given open covering of X and let

$h : I \to U_\lambda \cap U_\mu, h(0) = h(1) = x_0$

be a closed path. Let $\alpha \in \pi(U_\lambda, x_0)$ and $\beta \in \pi(U_\mu, x_0)$ denote the equivalence class of the loop h in the two different loops. Then, $\rho_\lambda(\alpha) = \rho_\mu(\beta)$

Proof

The set $U_v = U_\lambda \cap U_\mu$ also belongs to the covering by hypothesis, and h represents an element $\chi \in \pi(U_v, x_0)$. Then clearly,

$\alpha = \varphi_v \lambda(\chi)$

$\beta = \varphi_v \mu(\chi)$

Hence,

$\rho_\lambda(\alpha) = \rho_\lambda\ \varphi_{v\lambda}(\chi) = \rho_v(\chi)$

$\rho_\mu(\beta) = \rho_\mu\ \varphi_{v\mu}(\chi) = \rho_v(\chi)$

We can denote the element $\rho_\lambda(\alpha) = \rho_\mu(\beta) \in H$ by the notation $\rho(h)$; let

$\alpha_y = \rho[(g_{t-1,j} A_{ij})(g_{ij})^{-1}]$

$\beta_y = \rho[(g_{ij-1} B_{ij})(g_{ij})^{-1}]$

Here, $(g_{ij})^{-1}$ denote the path defined by $t \rightarrow g_{ij}$ (I – t)
Note that α_y and β_{ij} are both well-defined elements of H.
We assert that corresponding to each rectangle R_y, there is a relation of the following form in the group H:

$$\alpha_{i.t-1}\, \beta_{ij} = \beta_{i-1,j}\alpha_{ij} \tag{3}$$

To prove this, we note first the following equivalence between paths in $U_{\lambda(i,j)}$

$$A_{i.t-1}\, B_{ij} \sim B_{i-1,j}A_{ij}$$

We have the following equivalence between closed paths in U_λ(i,j).

$$g_{i-1,j-1}A_{tj-1}(g_{ij-1})^{-1}g_{ij-1}B_y(g_y)^{-1} \sim g_{i-1j-1}B_{i-1j}(g_{i-1j})^{-1}g_{i-1j}A_{ij}(g_{ij})^{-1} \tag{4}$$

If we take equivalence class in $\pi(U_{\lambda(I\ j)})$ of both sides and then apply the homomorphism $\rho_{\lambda(I\ j)}$, we obtain equation (3).
The next relation we need is

$$\prod_{t=1}^{m} \alpha_{t0} = \prod_{k=1}^{q} \rho_{\lambda k}(\beta_k) \tag{5}$$

Which is an easy consequence of requirement (b) that the points $\frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}$ be included in the set $[s_1 : 0\ I < m\}$ together with the definitions and constructions we have made. We have the relations.

$$\alpha_m = 1, \qquad 1 \leq I \leq m \tag{6}$$

$$\beta_{0\,I} = \beta_{mj} = 1, \quad 1 \leq j \leq n \tag{7}$$

The relations result from the fact that
F(s,1) = F(0,t)= F(1,t) = x_0 for any s, t $\in$ I.

In view of relation (5), we must prove

$$\prod_{t=1}^{m} \alpha_{t0} = 1 \tag{8}$$

We shall do this by using relations (3), (6) and (7)

First, we show that

$$\prod_{t=1}^{m} \alpha_{ij-1} = \prod_{k=1}^{q} \alpha_{ij-1} \tag{9}$$

For any integer j, $1 \leq j \leq n$. we have

$\alpha_{1\,j-1}\alpha_{2\,j-1}\ldots\alpha_{mj-1} = \alpha_{1\,j-1}\alpha_{2\,j-1}\ldots\alpha_{mj-1}\ \beta_{mj}$ by (7)
$= \alpha_{1\,j-1}\alpha_{2\,j-1}\ldots\alpha_{mj-1}\ \beta_{m-1j}\ \alpha_{mj}$ by (3)
$= \alpha_{1\,j-1}\alpha_{2\,j-1}\ldots\beta_{mj-1}\ \alpha_{m-1j}\ \alpha_{mj}$ by (3)
.
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$= \beta_{01}\, \alpha_{1\,j}\, \alpha_{2\,j} \ldots \alpha_{m-1\,j}\, \alpha_{m\,j}$ by (3)
$= \alpha_{1\,j}\, \alpha_{2\,j} \ldots \alpha_{m-1\,j}\, \alpha_{m\,j}$ by (7)

In all, we must apply (3) *m* times. If we now apply (9) with *j* = 1, 2, …, *n* in succession, we obtain

$$\prod_{t=1}^{m} \alpha_{t0} = \prod_{t=1}^{m} \alpha_m$$

But by use of (6),

$$\prod_{t=1}^{m} \alpha_{t0} = 1$$

This completes the proof of (8), and hence of Lemma 4.2.

5. Conclusion

In conclusion, the method of computing fundamental groups illustrates the general principle that calculations in algebraice topoloy usually work by piecing together a few pivotal example by means of general constructions or procedure.

We have also used the theorem to calculate the fundamental groups of the circle and of cell complexes and contracticle space and also of any compact surface.

Finally. Seifert-Van Kampen Theorem can be used to prove that any group is the fundamental group of some space using the definition of a simply connected space if it is path connected and satisfies $\pi_1(X) = 0$.

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