Common Fixed Point Theorems for Six Mappings in Fuzzy Metric Space

KAMAL WADHWA [1] and ASHLEKHA DUBEY [2]
[1]Govt. Narmada College Hoshangabad , M.P. India.
wadhwakamal68@gmail.com
[2]Sant Hirdaram Girls College Bhopal, M.P. India.
ashlekha.dubey@gmail.com

Abstract
The present paper deals with Some Common Fixed Point theorems for six occasionally weakly compatible (owc) mappings in Fuzzy Metric Space.
Keywords occasionally weakly compatible (owc) mappings, Fuzzy Metric Space.
Mathematics Subject Classification: 52H25, 47H10.

INTRODUCTION

Zadeh in 1965 introduced the concept of fuzzy set as a new way to represent vagueness in our everyday life. The development of fuzzy sets leads to develop a lot of literature regarding the theory of fuzzy sets and its application. A large number of renowned mathematicians worked with fuzzy sets in different branches of Mathematics, one such being the Fuzzy Metric Space. George and Verramani (1994) modified the concept of fuzzy metric space introduced by Kromosil and Michalek (1975). Many researchers have obtained common fixed point theorems for mappings satisfying different types of commutativity conditions.

In 1998, Jungck and Rhoades [1] introduced the notion of weakly compatible mappings and showed that compatible mappings are weakly compatible but not conversely. In 2006, Jungck and Rhoades [2] introduced occasionally weakly compatible mappings which is more general among the commutativity concepts. Jungck and Rhoades [2] obtained several common fixed point theorems using the idea of occasionally weakly compatible mappings. Several interesting and elegant results have been obtained by various authors in this direction few are namely Kamal wadhwa and Hariom Dubey [3], Pooja Sharma and R.S Chandel [6], Manish Jain, Sanjay Kumar and Satish Kumar [5], Rajesh Kumar Mishra and Sanjay Choudhary [7].

Preliminary Notes

Definition 1 [5]. A fuzzy set A in X is a function with domain X and values in [0, 1].
Definition 2 [5]. A mapping $*: [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm if ([0, 1], *) is an topological abelian monoid with unit 1 such that, $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for $a,b,c,d \in [0, 1]$.

Definition 3 [5]. A 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,\infty)$ satisfying the following conditions:
(a) $M(x, y, t) > 0$,
(b) $M(x, y, t) = 1$ if and only if $x = y$,
(c) $M(x, y, t) = M(y, x, t)$,
(d) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(e) $M(x, y, .): (0, \infty) \to (0, 1]$ is continuous, for $x ,y ,z \in X$ and $s ,t >0$
Then M is called a fuzzy metric on X. Then M(x, y, t) denotes the degree of nearness between x and y with respect to t.

Definition 4 [6]. Let (X, M, *) be a fuzzy metric space. Then
(a) A sequence $\{x_n\}$ in X is said to converges to x in X if for each $\in > 0$ and each $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x, t) > 1 - \in$ for all $n \geq n_0$.
(b) A sequence $\{x_n\}$ in X is said to be Cauchy if for each $\in >0$ and each $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \in$, for all $n, m \geq n_0$.
(c) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Example 1. Let (X, d) is a metric space. Define $a * b = ab$ or $a * b = \min(a,b)$ for all $x, y \in X$ and $t > 0$.
$$M(x, y, t) = t/t + d(x, y)$$
Then (X, M, *) is a fuzzy metric space and the fuzzy metric M induced by the metric d is often referred to as the standard fuzzy metric.

Lemma1. Let (X, M, $*$) be a Fuzzy Metric Space .If there exists q∈ (0, 1) such that M(x, y, qt)> M(x, y, t) for all x, y ∈ X and t>0, then x=y.
Lemma2. Let X is a set f and g is occasionally weakly compatible (owc) self-maps of X .If f and g have a unique point of coincidence w=fx=gx, then w is the unique common fixed point of f and g.
Definition 5 [6 **].** Let X is a set self-maps of X. A point in X is called a coincidence point of f and g if fx = gx. We shall call w = fx = gx a point of coincidence of f and g.

Definition 6 [6**].** A pair of maps S and T is called weakly compatible pair if they commute at coincidence points. The concept occasionally weakly compatible is introduced by Thagafi and Shahzad (2008). It is stated as follows.

Definition 7 [6 **].** Two self maps f and g of a set X are occasionally weakly compatible (owc) if there is a point x in X which is a coincidence point of f and g at which f and g commute. Thagafi and Shahzad (2008) shows that occasionally weakly is weakly compatible but converse is not true.

Example 2. Let R be the usual metric space. Define S, T: R→ R by Sx = 2x and Tx = x^2 for all x ∈ R. Then Sx = Tx for x = 0, 2 but ST0 = TS0 and ST2 ≠ TS2.

MAIN RESULTS

Theorem 1: Let (X, M, $*$) be a Fuzzy Metric Space and Let A, B, T, S, P, Q be self-Mappings of X. Let the pairs {AB, P} and {TS,Q} are Occasionally Weakly Compatible (owc), such that

$$M(ABx,TSy,qt) \geq Ø(Min\{M(Px,Qy,t),1/2[M(ABx,Px,t)+M(TSy,Qy,t)], 1/2[M(ABx,Qy,t)+ M(TSy,Px,t)]\}) \quad (1.1)$$

For all x, y ∈ X and Ø: [0, 1] → [0, 1] such that Ø(t) >t for all 0<t<1,then there exists a unique common fixed point of AB,TS,P and Q Furthermore, if the pairs (A, B) and (T, S) are the commuting pair of mappings then A, B, T, S, P and Q have a common fixed point.
Proof : Let the pairs {AB, P} and {TS,Q} be owc, so there exists points x, y ∈ X such that ABx=Px and TSy=Qy. We claim that ABx=TSy.
By (1.1)

$$\begin{aligned} M(ABx,TSy,qt) &\geq Ø(Min \{M(Px,Qy,t),1/2[M(ABx,Px,t)+ M(TSy,Qy,t)], 1/2[M(ABx,Qy,t)+ M(TSy,Px,t)]\}) \\ &= Ø(Min\{M(Px,Qy,t),1/2[M(ABx,ABx,t)+M(TSy,TSy,t)], 1/2[M(ABx,Qy,t)+ M(Qy,ABx,t)]\}) \\ &= Ø (Min \{M(Px,Qy,t),1/2[1+1],1/2[2M(ABx,Qy,t)]\}) \\ &= Ø (Min \{M(ABx,TSy,t),1,[M(ABx,TSy,t)]\}) \\ &= Ø (\{M(ABx,TSy,t)\}) \\ &> M (ABx,TSy,t) \end{aligned}$$

Hence ABx=TSy

Therefore ABx=Px = TSy=Qy (1.2)

Moreover if there is another point of Coincidence z, such that ABz=Pz, then using (1.2)
We get ,ABz=Pz=TSy=Qy . (1.3)
Now from (1.2) & (1.3) we get ABz = ABx, therefore z = x.
Hence w = ABx = Px for w ∈ X is the unique point of coincidence of AB and P by Lemma 2, w is the unique common fixed point of AB and P, i.e., ABw = Pw = w.
Similarly, there is a unique common fixed point u ∈ X Such that TSu = Qu= u.
Suppose that u ≠ w then by using condition (1.1), we get

$$\begin{aligned} M(w, u, qt) &= M(ABw, TSu, qt) \geq Ø(Min \{M(Pw, Qu, t)\} ,1/2[M(ABw, Pw, t) + M(TSu,Qu,t)], 1/2[M(ABw,Qu,t)+ M(TSu,Pw,t)]\}) \\ &= Ø(Min \{M(w,u,t),1/2[M(w,w,t)+ M(u,u,t)], 1/2[M(w,u,t)+ M(u,w,t)]\}) \\ &= Ø(Min \{M(w,u,t),1/2[1+1], 1/2[2M(w,u,t)]\}) \\ &= Ø(Min \{M(w,u,t),1,M(w,u,t)]\}) \\ &= Ø(\{M(w,u,t)\}) \\ &> M(w,u,t). \end{aligned}$$

By lemma 1, we have a w=u.
Hence w is a common fixed point of AB, TS, P, Q.
Finally we need to show that w is only the common fixed point of mappings A, B, T, S, P and Q. The pairs (A, B), (T, S) are commuting pairs therefore,

Aw=A(ABw)=A(BAw)=AB(Aw), which gives Aw=w.
Also Bw=B(ABw)=BA(Bw)=AB(Bw), which gives Bw=w
Similarly Tw=w and Sw=w.
Hence A, B, T, S, P and Q have a unique common Fixed Point.

Theorem 2: Let (X, M, $*$) be a Fuzzy Metric Space and Let A, B, T, S, P, Q be self Mappings of X such that:
(i) {AB, P} and {TS, Q} are Occasionally Weakly Compatible (owc) mappings
(ii) $M(ABx,TSy,qt) \geq Ø(Min\{M(Px,Qy,t), M(ABx,Qy,t), M(TSy,Px,t), 1/2[M(ABx,Px,t) + M(TSy,Qy,t)]\})$ (2.1)
For all x,y $\in$ X and $Ø:[0,1] \rightarrow [0,1]$ such that $Ø(t) > t$ for all $0<t<1$, then there exists a unique common fixed point of AB,TS,P and Q Furthermore, if the pairs (A, B) and (T, S) are the commuting pair of mappings then A, B, T, S, P and Q have a common fixed point.
Proof: Let the pairs {AB, P} and {TS,Q} be owc, so there exists points x, y $\in$ X, such that ABx=Px and TSy=Qy. We claim that ABx=TSy.
By (2.1)

$$\begin{aligned} M(ABx,TSy,qt) &\geq Ø(Min\{M(Px,Qy,t), M(ABx,Qy,t), M(TSy,Px,t), 1/2[M(ABx,Px,t) + M(TSy,Qy,t)]\}) \\ &= Ø(Min\{M(ABx,TSy,t), M(ABx,TSy,t), M(TSy,ABx,t), 1/2[M(ABx,ABx,t) + M(TSy,TSy,t)]\}) \\ &= Ø(Min\{M(ABx,TSy,t), M(ABx,TSy,t), M(TSy,ABx,t), 1/2[1+1]\}) \\ &= Ø(Min\{M(ABx,TSy,t), M(ABx,TSy,t), M(TSy,ABx,t), 1\}) \\ &= Ø(\{M(ABx,TSy,t)\}) \\ &> M(ABx,TSy,t) \end{aligned}$$

Hence, by lemma 1, ABx=TSy
Therefore ABx=Px = TSy=Qy (2.2)
Moreover if there is another point of coincidence z, such that ABz=Pz ,then using (2.2) we get,

$$ABz=Pz=TSy=Qy \quad (2.3)$$

Now from (2.2) & (2.3) we get ABz= ABx, therefore z=x.
Hence w=ABx=Px for w $\in$ X is the unique point of coincidence of AB and P by Lemma 2, w is the unique common fixed point of AB and P, i.e., ABw = Pw = w.
Similarly, there is a unique common fixed point u $\in$ X such that u = TSu = Qu
Suppose that $u \neq w$
Then by using condition (2.1) we get

$$\begin{aligned} M(w,u,qt) &= M(ABw,TSu,qt) \geq Ø(Min\{M(Pw,Qu,t), M(ABw,Qu,t), M(TSu,Pw,t), 1/2[M(ABw,Pw,t) + M(TSu,Qu,t)]\}) \\ &= Ø(Min\{M(w,u,t), M(w,u,t), M(u,w,t), 1/2[M(w,w,t) + M(u,u,t)]\}) \\ &= Ø(Min\{M(w,u,t), M(w,u,t), M(u,w,t), 1/2[1+1]\}) \\ &= Ø(Min\{M(w,u,t), M(w,u,t), M(u,w,t), 1\}) \\ &= Ø(\{M(w,u,t)) \\ &> M(w,u,t). \end{aligned}$$

By lemma 1, we have a w=u.
Hence w is a common fixed point of AB, TS, P, and Q.
Finally we need to show that w is only the common fixed point of mappings A, B, T, S, P and Q.
The pairs (A, B), (T, S) are commuting pairs therefore
Aw=A(ABw)=A(BAw)=AB(Aw), which gives Aw=w.
Also Bw=B(ABw)=BA(Bw)=AB(Bw), which gives Bw=w
Similarly Tw=w and Sw=w.
Hence A, B, T, S, P and Q have a unique common fixed point.

Remark :
1. In 2013, Pooja Sharma and R. S. Chandel [6] claimed to prove some following fixed point theorem for six occasionally weakly compatible mappings:
Theorem: Let (X, M, $*$) be a complete fuzzy metric Space and Let A, B, S, T, P, Q be self- Mappings of X. Let the pairs {P, ST} and {Q, AB} are (owc), if there exists q $\in$ (0,1) such that

$$M(Px,Qy,qt) \geq Ø[(min\{M(STx,ABy,t), M(STx,Px,t)\} * min\{M(Qy,ABy,t), M(Px,ABy,t), M(Qy,STx,t)\}]$$

For all x, y $\in$ X and $Ø: [0,1] \rightarrow [0,1]$ such that $Ø(t) > t$ for all $0<t<1$, then there exists a unique common fixed point of A,B,S,T,P and Q.

In the proof of the theorem, they found common fixed point of four mappings namely ST, AB, P & Q. Following the proof of theorem 1 with above contractive condition, one can obtain common fixed point theorem for six mappings.

2. Our result generalizes the results of K.Jha, M.Imdad and U.Rajopadhyaya [4] in the Fuzzy Metric Space.

References

[1] G.Jungck and B.E.Rhodes, ***"Fixed Point for set valued function without continuity"***, Indian J.Pure Appl.Math., 293(3)(1998)227-238.

[2] G.Jungck and B.E.Rhodes, ***"Fixed Point for occasionally weakly compatible mappings"***, Fixed Point Theory, 7(2006)280-296.

[3] Kamal Wadhwa and Hariom Dubey, ***" On Fixed Point Theorems for Four Mappings in Fuzzy Metric Spaces"*** ,IMACST:VOLUME 2 NUMBER 1MAY 2011,5-8.

[4] K.Jha,M.Imdad and U.Rajopadhyaya, ***"weakly compatible mappings in semi metric space "***, Annals of pure and Applied Mathematics,Vol.5,No .2,2014,153-157.

[5] Manish Jain, Sanjay Kumar and Satish Kumar, "***Fixed Point Theorems for Occasionally Weakly Compatible Maps in Fuzzy Metric Spaces***" ,International Journal Of Computer Applications(0975-8887),volume 19-No.9,April 2011.

[6] Pooja Sharma and R S Chandel, ***"Fixed Point Theorems for weakly compatible mappings in fuzzy metric spaces"*** .Fasciculi Mathematici , nr 51, 2013(141-148).

[7] Rajesh Kumar Mishra and Sanjay Choudhary, "***On Fixed Point Theorems in Fuzzy Metric Spaces***",IMACST:VOLUME 1 NUMBER 1 DECEMBER 2010,45-47.