Extension of Some Common Fixed Point Theorems of Integral Type Mappings in Hilbert Space

[1] Pravin B. Prajapati, [2] Ramakant Bhardwaj, Piyush Bhatnagar[3]
[1]S.P.B.Patel Engineering College, Linch
[1] The Research Scholar of Sai nath University, Ranchi (Jharkhand)
[2]Truba Institute of Engineering & Information Technology, Bhopal (M.P)
[3]Department of Mathematics, Govt. M.L.B.College Bhopal
E-mail: rkbhardwaj100@gmail.com

ABSTRACT
The object of this paper is to obtain common fixed point theorems for two continuous mapping s of a Hilbert space for integral type mapping**.**
Key wards: common fixed point, rational expression, Hilbert space .uniqueness
AMS Subject Classification: 47H10, 54H25

2. Introduction and Preliminaries

Impact of fixed point theory in different branches of mathematics and its applications is immense. The first result on fixed points for Contractive type mapping was the much celebrated Banach's contraction principle by S. Banach [13] in 1922. In the general setting of complete metric space, this theorem runs as the follows

Theorem 2.1 (Banach's contraction principle) Let (X,d) be a complete metric space, c∈(0,1) and f: X→X be a mapping such that for each x, y ∈X,
d (fx,fy) ≤ cd (x,y) Then f has a unique fixed point a ∈X, such that for each
x ∈ X, $\lim_{n\to\infty} f^n(x) = a.$

After the classical result, Kannan [11] gave a subsequently new contractive mapping to prove the fixed point theorem. Since then a number of mathematicians have been worked on fixed point theory dealing with mappings satisfying various type of contractive conditions.

In 2002, A. Branciari [1] analysed the existence of fixed point for mapping f defined on a complete metric space (X,d) satisfying a general contractive condition of integral type.

Theorem 2.2 (Branciari) Let (X,d) be a complete metric space ,c $\in (0,1)$ and let f : X→ X be a mapping such that for each x, y ∈X,
$\int_0^{d\,(fx,fy)} \emptyset(t)dt \le c \int_0^{d\,(x,y)} \emptyset(t)dt$ where $\emptyset$:[0,+ ∞) →[0,+ ∞) is a Lebesgue integrable mapping which is summable on each compact subset of [0,+ ∞) , non negative ,and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \emptyset(t)dt$, then f has a unique fixed point a ∈X, such that for each x ∈ X,
$\lim_{n\to\infty} f^n(x) = a.$

After the paper of Branciari, a lot of research works have been carried out on generalizing contractive condition of integral type for different contractive mappings satisfying various known properties. A fine work has been done by Rhoades [2] extending the result of Branciari by replacing the condition [1.2] by the following

$$\int_0^{d\,(fx,fy)} \emptyset(t)dt \le \int_0^{\max\left\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}\right\}} \emptyset(t)dt.$$

Koparde and Waghmode [9] extended the result of Jungck[6] and Fisher[3] to common fixed points.Here we proceed to extend and sharpen the main result of
Koparde and Waghmode [9] by modifying their procedure.
Jungck [6] obtained the following
Theorem 2.3: Let f be a continuous mapping of a complete metric space (X,d) into itself. Then f has a fixed point in X if there exists $\alpha \in (0,1)$ and a mapping g : X→ X which commutes with f and satisfies
g(X) ⊂ f(X) and d(g(x),g(y)) ≤ α d(f(x),f(y)) ∀ x,y ∈ X

Indeed, f and g have a unique common fixed point.
Fisher [3] extended and modified the above result in the form of
Theorem 2.4: Let S and T be continuous mapping of complete metric space (X,d) into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping A of X into SX$\cap TX$, which commute with S and T and satisfies the inequality

d(Ax,Ay) ≤ α d(Sx,Sy) $\forall$ x,y $\in X$

Where $0 < \alpha < 1$. Indeed S, T, and A then have a unique common fixed point.

Extending further, Koparde and Waghmode [9] obtained the following in setting of Hilbert space.
Theorem 2.5: Let S and T be continuous mapping of Hilbert space (X,d) into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping A of X into SX$\cap TX$, which commute with S and T and satisfies the inequality

$$\|Ax - Ay\| \leq \alpha \, \|Ax - Sx\| + \beta \|Ay - Ty\| + \gamma \|Sx - Ty\|$$

For all x,y in X , where α, β, γ are non- negative reals with $0 < \alpha + \beta + \gamma < 1$. Indeed, S,T and A then have a unique common fixed point.
The aim of this paper is to generalize and modified the above result in integral type mappings.

3. MAIN RESULTS

Theorem 3.1:
Let S and T be continuous mapping of Hilbert space (X,d) into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping A of X into SX$\cap TX$, which commute with S and T and satisfies the inequality

$$\int_0^{\|Ax-Ay\|^2} \emptyset(t)dt \leq \alpha \int_0^{\|Ax-Sx\|^2} \emptyset(t)dt + \beta \int_0^{\|Ay-Ty\|^2} \emptyset(t)dt + \gamma \int_0^{\|Ax-Sy\|^2} \emptyset(t)dt + \delta \int_0^{\|Ax-Ty\|^2} \emptyset(t)dt + \eta \int_0^{\max\{\|Ay-Sy\|^2, \|Ay-Tx\|^2\}} \emptyset(t)dt$$

For all x,y in X ; where α , $\beta, \gamma, \delta, \eta$ are non-negative reals with $0 < \alpha + \beta + 2\eta + \gamma + \delta < 1$. Also $\emptyset$:[0,+ ∞) $\rightarrow$[0,+ ∞) is a Lebesgue integrable mapping which is summable on each compact subset of [0,+ ∞) , non- negative ,and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \emptyset(t)dt$, then S,T and A then have a unique common fixed point.

Proof: First of all we prove that the existence of such a mapping A is necessary. For this suppose ,Sz=z=Tz for some z in X.

Let us define a mapping A of X into X by Ax=z for all x in x. Then clearly , A is continuous mapping of X into SX$\cap TX$. Since, Sx, Tx $\in X$, for all
x $\in X$ and Ax=z for all x$\in X$, we get
ASx = z ,SAx =Sz =z ,ATx=z, TAx=Tz=z.

Hence, A commutes with S and T. Now for any α , $\beta, \gamma, \delta, \eta$ with
$0 < \alpha + \beta + 2\eta + \gamma + \delta < 1$, it is observe that

$$\int_0^{\|Ax-Ay\|^2} \emptyset(t)dt \le \alpha \int_0^{\|Ax-Sx\|^2} \emptyset(t)dt + \beta \int_0^{\|Ay-Ty\|^2} \emptyset(t)dt$$
$$+ \gamma \int_0^{\|Ax-Sy\|^2} \emptyset(t)dt + \delta \int_0^{\|Ax-Ty\|^2} \emptyset(t)dt$$
$$+ \eta \int_0^{\max\{\|Ay-Sy\|^2, \|Ay-Tx\|^2\}} \emptyset(t)dt$$

Which gives

$$0 \le \alpha \int_0^{\|z-Sx\|^2} \emptyset(t)dt + \beta \int_0^{\|z-Ty\|^2} \emptyset(t)dt$$
$$+ \gamma \int_0^{\|z-Sy\|^2} \emptyset(t)dt + \delta \int_0^{\|z-Ty\|^2} \emptyset(t)dt$$
$$+ \eta \int_0^{\max\{\|z-Sy\|^2, \|z-Tx\|^2\}} \emptyset(t)dt$$

For all x,y in X. This proves the existence of such a mapping A is necessary.
To prove sufficient part, sequence $\{x_n\}$ is constructed as follows. Let $x_0 \in X$ be an arbitrary point .

Since AX $\subset$ SX , we choose a point x_1 in X such that S $x_1 = A\, x_0$.

Also AX $\subset$ TX and hence we can choose $x_2 \in X$ such that T$x_2 = A\, x_1$.

Continuing in this way, we get a sequence $\{x_n\}$ as follows :

S $x_{2n-1} = A\, x_{2n-2}$, T$x_{2n} = A\, x_{2n-1}$ n = 1,2,3 . ………

We proceed to show that $\{Ax_n\}$ is a Cauchy sequence .

For this we have the inequality,

$$\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \emptyset(t)dt = \int_0^{\|Ax_{2n-1}-Ax_{2n}\|^2} \emptyset(t)dt$$
$$\le \alpha \int_0^{\|Ax_{2n-1}-Sx_{2n-1}\|^2} \emptyset(t)dt + \beta \int_0^{\|Ax_{2n}-Tx_{2n}\|^2} \emptyset(t)dt$$
$$+ \gamma \int_0^{\|Ax_{2n-1}-Sx_{2n}\|^2} \emptyset(t)dt + \delta \int_0^{\|Ax_{2n-1}-Tx_{2n}\|^2} \emptyset(t)dt$$
$$+ \eta \int_0^{\max\{\|Ax_{2n}-Sx_{2n}\|^2, \|Ax_{2n}-Tx_{2n-1}\|^2\}} \emptyset(t)dt$$

$$\le \alpha \int_0^{\|Ax_{2n-1}-Ax_{2n-2}\|^2} \emptyset(t)dt + \beta \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \emptyset(t)dt$$
$$+ \gamma \int_0^{\|Ax_{2n-1}-Ax_{2n-1}\|^2} \emptyset(t)dt + \delta \int_0^{\|Ax_{2n-1}-Ax_{2n-1}\|^2} \emptyset(t)dt$$
$$+ \eta \int_0^{\max\{\|Ax_{2n}-Ax_{2n-1}\|^2, \|Ax_{2n}-Ax_{2n-2}\|^2\}} \emptyset(t)dt \quad \text{--------} \quad (3.1.1)$$

Case: 1 If
$\max\{\|Ax_{2n}-Ax_{2n-1}\|^2, \|Ax_{2n}-Ax_{2n-2}\|^2\} = \|Ax_{2n}-Ax_{2n-1}\|^2$
From (3.1.1)

$$\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \phi(t)dt \leq \alpha \int_0^{\|Ax_{2n-1}-Ax_{2n-2}\|^2} \phi(t)dt + (\beta+\eta)\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \phi(t)dt$$

$$\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \phi(t)dt \leq \frac{\alpha}{1-\beta-\eta}\int_0^{\|Ax_{2n-1}-Ax_{2n-2}\|^2} \phi(t)dt \quad ----- (3.1.2)$$

Further, it is seen that

$$\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^2} \phi(t)dt = \int_0^{\|Ax_{2n}-Ax_{2n+1}\|^2} \phi(t)dt$$

$$\leq \alpha \int_0^{\|Ax_{2n}-Sx_{2n}\|^2} \phi(t)dt + \beta \int_0^{\|Ax_{2n+1}-Tx_{2n+1}\|^2} \phi(t)dt + \gamma \int_0^{\|Ax_{2n}-Sx_{2n+1}\|^2} \phi(t)dt + \delta \int_0^{\|Ax_{2n}-Tx_{2n+1}\|^2} \phi(t)dt + \eta \int_0^{\|Ax_{2n+1}-Sx_{2n+1}\|^2} \phi(t)dt$$

$$= \alpha \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \phi(t)dt + \beta \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^2} \phi(t)dt + \eta \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^2} \phi(t)dt$$

This implies that

$$\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^2} \phi(t)dt \leq \frac{\alpha}{1-\beta-\eta}\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \phi(t)dt$$

$$= \lambda_1 \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \phi(t)dt \quad ----- (3.1.3)$$

Where $\lambda_1 = \frac{\alpha}{1-\beta-\eta}$

Case: 2 If

$$\max\{\|Ax_{2n}-Ax_{2n-1}\|^2, \|Ax_{2n}-Ax_{2n-2}\|^2\} = \|Ax_{2n}-Ax_{2n-2}\|^2$$

From (3.1.1)

$$\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \phi(t)dt \leq \alpha \int_0^{\|Ax_{2n-1}-Ax_{2n-2}\|^2} \phi(t)dt + \beta \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \phi(t)dt + \eta \int_0^{\|Ax_{2n}-Ax_{2n-2}\|^2} \phi(t)dt$$

$$\leq \alpha \int_0^{\|Ax_{2n-1}-Ax_{2n-2}\|^2} \phi(t)dt + \beta \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \phi(t)dt + \eta \left[\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \phi(t)dt + \int_0^{\|Ax_{2n-1}-Ax_{2n-2}\|^2} \phi(t)dt\right]$$

$$\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \phi(t)dt \leq \frac{\alpha+\eta}{1-\beta-\eta}\int_0^{\|Ax_{2n-1}-Ax_{2n-2}\|^2} \phi(t)dt \quad ----- (3.1.4)$$

Further, it is seen that

$$\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^2} \phi(t)dt = \int_0^{\|Ax_{2n}-Ax_{2n+1}\|^2} \phi(t)dt$$

$$\leq \alpha \int_0^{\|Ax_{2n}-Sx_{2n}\|^2} \emptyset(t)dt + \beta \int_0^{\|Ax_{2n+1}-Tx_{2n+1}\|^2} \emptyset(t)dt$$
$$+ \gamma \int_0^{\|Ax_{2n}-Sx_{2n+1}\|^2} \emptyset(t)dt + \delta \int_0^{\|Ax_{2n}-Tx_{2n+1}\|^2} \emptyset(t)dt$$
$$+ \eta \int_0^{\|Ax_{2n+1}-Tx_{2n}\|^2} \emptyset(t)dt$$
$$= \alpha \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \emptyset(t)dt + \beta \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^2} \emptyset(t)dt$$
$$+ \eta \int_0^{\|Ax_{2n+1}-Ax_{2n-1}\|^2} \emptyset(t)dt$$
$$\leq \alpha \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \emptyset(t)dt + \beta \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^2} \emptyset(t)dt$$
$$+ \eta \left[\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^2} \emptyset(t)dt + \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \emptyset(t)dt \right]$$

This implies that

$$\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^2} \emptyset(t)dt \leq \frac{\alpha+\eta}{1-\beta-\eta} \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \emptyset(t)dt$$

$$= \lambda_2 \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \emptyset(t)dt \quad ----- \text{(3.1.5)}$$

Where $\lambda_2 = \frac{\alpha+\eta}{1-\beta-\eta}$

From both cases

Taking $\lambda = \max\{\lambda_1, \lambda_2\}$

$0 < \lambda < 1$ as we have

$0 < \alpha + \beta + 2\eta < 1$

Now, from the both cases

$$\int_0^{\|Ax_{n+1}-Ax_n\|^2} \emptyset(t)dt \leq \lambda \int_0^{\|Ax_n-Ax_{n-1}\|^2} \emptyset(t)dt$$
$$\leq \lambda^2 \int_0^{\|Ax_{n-1}-Ax_{n-2}\|^2} \emptyset(t)dt$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\leq \lambda^n \int_0^{\|Ax_1-Ax_0\|^2} \emptyset(t)dt$$

For large n. Now, it can be seen that $\{Ax_n\}$ is a Cauchy sequence and so it has a limit x in X. Since sequences $\{Sx_{2n+1}\}$ and $\{Tx_{2n}\}$ are subsequences of $\{Ax_n\}$, they have the same limit z. As S and A are commuting mapping, we can have

$$Sz = \lim_{n\to\infty} S\,Ax_{2n+1}$$
$$= \lim_{n\to\infty} A\,S\,x_{2n+1}$$
$$= Az$$

Similarly, we get Tz = Az

This gives us,

Tz = Az = Sz

It comes out that

$$\int_0^{\|Az-AAz\|^2} \emptyset(t)dt \le \alpha \int_0^{\|Az-Sz\|^2} \emptyset(t)dt + \beta \int_0^{\|AAz-TAz\|^2} \emptyset(t)dt + \gamma \int_0^{\|Az-SAz\|^2} \emptyset(t)dt + \delta \int_0^{\|Az-TAz\|^2} \emptyset(t)dt + \eta \int_0^{\max\{\|AAz-SAz\|^2, \|AAz-Tz\|^2\}} \emptyset(t)dt$$

Commutatively of A with S and T gives the following by

$\|AAz - TAz\| = \|AAz - ATz\| = \|AAz - AAz\| = 0$

And

$\|Az - SAz\| = \|Az - AAz\|$ and $\|Az - TAz\| = \|Az - AAz\|$

Also ,

$\|AAz - SAz\| = \|AAz - ASz\| = \|AAz - AAz\| = 0$

Resorting to these, we arrive at

$$\int_0^{\|Az-AAz\|^2} \emptyset(t)dt \le (\gamma + \delta + \eta) \int_0^{\|Az-AAz\|^2} \emptyset(t)dt$$

Since $\alpha + \beta + 2\eta + \gamma + \delta < 1$, we have that Az = AAz .

Finally, putting Az = z_1 , we have Az_1 = Az = AAz = z_1

Similarly, Tz_1 = TAz = ATz =AAz =Az = z_1 ,

Sz_1 = z_1 .

So , z_1 is a fixed point of S ,T and A.

Next , to show uniqueness of this Common fixed point , let us suppose that z_2 ia also a common fixed point of S ,T and A other than z_1 .

Then Sz_2 = z_2 , Tz_2 = z_2 , Az_2 = z_2 , and also $\|z_1 - z_2\| \neq 0$.

Hence, it follows that

$$\int_0^{\|z_1-z_2\|^2} \emptyset(t)dt = \int_0^{\|Az_1-Az_2\|^2} \emptyset(t)dt \le \alpha \int_0^{\|Az_1-Sz_1\|^2} \emptyset(t)dt + \beta \int_0^{\|Az_2-Tz_2\|^2} \emptyset(t)dt + \gamma \int_0^{\|Az_1-Sz_2\|^2} \emptyset(t)dt + \delta \int_0^{\|Az_1-Tz_2\|^2} \emptyset(t)dt + \eta \int_0^{\max\{\|Az_2-Sz_2\|^2, \|Az_2-Tz_1\|^2\}} \emptyset(t)dt$$

Implies

$$\int_0^{\|z_1-z_2\|^2} \emptyset(t)dt \le (\gamma + \delta + \eta) \int_0^{\|z_1-z_2\|^{p/q}} \emptyset(t)dt$$

Which is a contradiction as $\alpha + \beta + 2\eta + \gamma + \delta < 1$.

Thus occurs the uniqueness.

Theorem 3.2:

Let S and T be continuous mapping of Hilbert space (X,d) into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping A of X into SX$\cap TX$, which commute with S and T and satisfies the inequality

$$\int_0^{\|Ax-Ay\|^2} \emptyset(t)dt \le \alpha \int_0^{\|Ax-Sx\|\,\|Ay-Sy\|} \emptyset(t)dt + \beta \int_0^{\|Ax-Tx\|\,\|Ay-Ty\|} \emptyset(t)dt + \gamma \int_0^{\|x-y\|^2} \emptyset(t)dt + \delta \int_0^{\|Sx-Ty\|^2} \emptyset(t)dt + \eta \int_0^{\|Sx-Tx\|^2} \emptyset(t)dt$$

For all x,y in X ; where α , $\beta, \gamma, \delta, \eta$ are non-negative reals with $0 < \alpha + \beta + 2\eta + \gamma + \delta < 1$. Also $\emptyset$:[0,+ ∞) $\rightarrow$[0,+ ∞) is a Lebesgue integrable mapping which is summable on each compact subset of [0,+ ∞) , non- negative ,and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \emptyset(t)dt$, then S,T and A then have a unique common fixed point .

Proof: First of all we prove that the existence of such a mapping A is necessary. For this suppose ,Sz=z=Tz for some z in X.

Let us define a mapping A of X into X by Ax=z for all x in x. Then clearly , A is continuous mapping of X into SX$\cap TX$. Since, Sx, Tx $\in X$, for all

x $\in X$ and Ax=z for all x$\in X$, we get

ASx = z , SAx =Sz =z ,ATx=z,TAx=Tz=z.

Hence, A commutes with S and T. Now for any α , $\beta, \gamma, \delta, \eta$ with

$0 < \alpha + \beta + 2\eta + \gamma + \delta < 1$, it is observe that

$$\int_0^{\|Ax-Ay\|^2} \emptyset(t)dt \le \alpha \int_0^{\|Ax-Sx\|\,\|Ay-Sy\|} \emptyset(t)dt + \beta \int_0^{\|Ax-Tx\|\,\|Ay-Ty\|} \emptyset(t)dt + \gamma \int_0^{\|x-y\|^2} \emptyset(t)dt + \delta \int_0^{\|Sx-Ty\|^2} \emptyset(t)dt + \eta \int_0^{\|Sx-Tx\|^2} \emptyset(t)dt$$

This gives,

$$0 \le \alpha \int_0^{\|Ax-Sx\|\,\|Ay-Sy\|} \emptyset(t)dt + \beta \int_0^{\|Ax-Tx\|\,\|Ay-Ty\|} \emptyset(t)dt + \gamma \int_0^{\|x-y\|^2} \emptyset(t)dt + \delta \int_0^{\|Sx-Ty\|^2} \emptyset(t)dt + \eta \int_0^{\|Sx-Tx\|^2} \emptyset(t)dt$$

For all x,y in X. This proves the existence of such a mapping A is necessary.

To prove sufficient part, sequence $\{x_n\}$ is constructed as follows. Let $x_0 \in X$ be an arbitrary point.

Since AX $\subset$ SX , we choose a point x_1 in X such that S $x_1 = A\,x_0$.

Also AX $\subset$ TX and hence we can choose $x_2 \in X$ such that T$x_2 = A\,x_1$.

Continuing in this way, we get a sequence $\{x_n\}$ as follows :

S $x_{2n-1} = A\,x_{2n-2}$, T$x_{2n} = A\,x_{2n-1}$ n = 1,2,3 . ………

We proceed to show that $\{Ax_n\}$ is a Cauchy sequence .

For this we have the inequality,

$$\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^2} \emptyset(t)dt \le \alpha \int_0^{\|Ax_{2n+1}-Sx_{2n+1}\|\|Ax_{2n}-Sx_{2n}\|} \emptyset(t)dt + \beta \int_0^{\|Ax_{2n+1}-Tx_{2n+1}\|\|Ax_{2n}-Tx_{2n}\|} \emptyset(t)dt + \gamma \int_0^{\|x_{2n+1}-x_{2n}\|^2} \emptyset(t)dt + \delta \int_0^{\|Sx_{2n+1}-Tx_{2n}\|^2} \emptyset(t)dt + \eta \int_0^{\|Sx_{2n+1}-Tx_{2n+1}\|^2} \emptyset(t)dt$$

$$\le \alpha \int_0^{\|Ax_{2n+1}-Ax_{2n}\|\|Ax_{2n}-Ax_{2n-1}\|} \emptyset(t)dt + \beta \int_0^{\|Ax_{2n+1}-Ax_{2n}\|\|Ax_{2n}-Ax_{2n-1}\|} \emptyset(t)dt + \gamma \int_0^{\|x_{2n+1}-x_{2n}\|^2} \emptyset(t)dt + \delta \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \emptyset(t)dt + \eta \int_0^{\|Ax_{2n}-Ax_{2n}\|^2} \emptyset(t)dt$$

$$\le (\alpha+\beta) \int_0^{\left[\frac{\|Ax_{2n+1}-Ax_{2n}\|^2}{2}+\frac{\|Ax_{2n}-Ax_{2n-1}\|^2}{2}\right]} \emptyset(t)dt + \gamma \int_0^{\|x_{2n+1}-x_{2n}\|^2} \emptyset(t)dt + \delta \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \emptyset(t)dt + \eta \int_0^{\|Ax_{2n}-Ax_{2n}\|^2} \emptyset(t)dt$$

This gives

$$2\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^2} \emptyset(t)dt \le (\alpha+\beta) \int_0^{[\|Ax_{2n+1}-Ax_{2n}\|^2+\|Ax_{2n}-Ax_{2n-1}\|^2]} \emptyset(t)dt + 2\gamma \int_0^{\|x_{2n+1}-x_{2n}\|^2} \emptyset(t)dt + 2\delta \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \emptyset(t)dt + 2\eta \int_0^{\|Ax_{2n}-Ax_{2n}\|^2} \emptyset(t)dt$$

Resulting in

$$\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^2} \emptyset(t)dt \le \left(\frac{\alpha+\beta+2\,\delta}{2-\alpha-\beta}\right) \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \emptyset(t)dt$$

$$+\left(\frac{2\gamma}{2-\alpha-\beta}\right)\int_0^{\|x_{2n+1}-x_{2n}\|^2}\emptyset(t)dt \quad \text{------------ (3.2.1)}$$

Further, observe that

$$\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2}\emptyset(t)dt \le \alpha\int_0^{\|Ax_{2n}-Sx_{2n}\|\|Ax_{2n-1}-Sx_{2n-1}\|}\emptyset(t)dt$$
$$+\beta\int_0^{\|Ax_{2n}-Tx_{2n}\|\|Ax_{2n-1}-Tx_{2n-1}\|}\emptyset(t)dt$$
$$+\gamma\int_0^{\|x_{2n}-x_{2n-1}\|^2}\emptyset(t)dt+\delta\int_0^{\|Sx_{2n}-Tx_{2n-1}\|^2}\emptyset(t)dt$$
$$+\eta\int_0^{\|Sx_{2n}-Tx_{2n}\|^2}\emptyset(t)dt$$

Simplifying this using Young's inequality , we get

$$\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2}\emptyset(t)dt \le \left(\frac{\alpha+\beta+2\delta}{2-\alpha-\beta}\right)\int_0^{\|Ax_{2n-1}-Ax_{2n-2}\|^2}\emptyset(t)dt$$
$$+\left(\frac{2\gamma}{2-\alpha-\beta}\right)\int_0^{\|x_{2n}-x_{2n-1}\|^2}\emptyset(t)dt \quad \text{------------ (3.2.2)}$$

Since $\alpha+\beta+2\eta+\gamma+\delta<1$. We find that

$$\lambda_1=\frac{\alpha+\beta+2\delta}{2-\alpha-\beta},\ \lambda_2=\frac{2\gamma}{2-\alpha-\beta}\in(0,1)$$

Suppose, now

$$\lambda=\max\{\lambda_1,\lambda_2\}$$

then $0<\lambda<1$.

From (3.2.1) and (3.2.2) ,we conclude that

$$\int_0^{\|Ax_{n+1}-Ax_n\|^2}\emptyset(t)dt \le \lambda\int_0^{[\|Ax_n-Ax_{n-1}\|^2+\|Ax_{n+1}-Ax_n\|^2]}\emptyset(t)dt$$
$$\le\lambda^2\int_0^{[\|Ax_{n-1}-Ax_{n-2}\|^2+\|Ax_n-Ax_{n-1}\|^2]}\emptyset(t)dt$$
$$+\lambda\int_0^{[\|Ax_{n+1}-Ax_n\|^2]}\emptyset(t)dt$$

$$\le\lambda^n\int_0^{[\|Ax_1-Ax_0\|^2+\|Ax_2-Ax_1\|^2]}\emptyset(t)dt+\varepsilon,$$

For large n.

. Now , it can be seen that $\{Ax_n\}$ is a Cauchy sequence and so it has a limit x in X. Since sequences $\{Sx_{2n+1}\}$ and $\{Tx_{2n}\}$ are sub-sequences of $\{Ax_n\}$, they have the same limit z. As S and A are commuting mapping, we can have

$$Sz=\lim_{n\to\infty}SAx_{2n+1}$$

$= \lim_{n\to\infty} A\, S\, x_{2n+1}$

$= Az$

Similarly, we get Tz = Az

This gives us ,

Tz = Az = Sz

It comes out that

$$\int_0^{\|Az-AAz\|^2} \emptyset(t)dt \le \alpha \int_0^{\|Az-Sz\|\|AAz-SAz\|} \emptyset(t)dt + \beta \int_0^{\|Az-Tz\|\|AAz-TAz\|} \emptyset(t)dt + \gamma \int_0^{\|z-Az\|^2} \emptyset(t)dt + \delta \int_0^{\|Sz-TAz\|^2} \emptyset(t)dt + \eta \int_0^{\|Sz-Tz\|^2} \emptyset(t)dt$$

Commutatively of A with S and T gives the following by

$\|AAz - TAz\| = \|AAz - ATz\| = \|AAz - AAz\| = 0$

And

$\|Az - SAz\| = \|Az - AAz\|$ and $\|Az - TAz\| = \|Az - AAz\|$

Also,

$\|AAz - SAz\| = \|AAz - ASz\| = \|AAz - AAz\| = 0$

Resorting to these, we arrive at

$$\int_0^{\|Az-AAz\|^2} \emptyset(t)dt \le \gamma \int_0^{\|z-Az\|^2} \emptyset(t)dt + \delta \int_0^{\|Az-AAz\|^2} \emptyset(t)dt$$

Since $\gamma + \delta < 1$, we must have Az = AAz

Finally, putting Az $= z_1$, we have Az_1 = Az = AAz = z_1

Similarly, Tz_1 = TAz = ATz =AAz =Az = z_1 ,

Sz_1 = z_1 .

So, z_1 is a fixed point of S ,T and A.

Next , to show uniqueness of this Common fixed point , let us suppose that z_2 ia also a common fixed point of S ,T and A other than z_1 .

Then Sz_2 = z_2 , Tz_2 = z_2 , Az_2 = z_2 , and also $\|z_1 - z_2\| \neq 0$.

Hence, it follows that

$$\int_0^{\|z_1-z_2\|^2} \emptyset(t)dt = \int_0^{\|Az_1-Az_2\|^2} \emptyset(t)dt$$

$$\le \alpha \int_0^{\|Az_1-Sz_1\|\|Az_2-Sz_2\|} \emptyset(t)dt + \beta \int_0^{\|Az_1-Tz_1\|\|Az_2-Tz_2\|} \emptyset(t)dt + \gamma \int_0^{\|z_1-z_2\|^2} \emptyset(t)dt + \delta \int_0^{\|Sz_1-Tz_2\|^2} \emptyset(t)dt + \eta \int_0^{\|Sz_1-Tz_1\|^2} \emptyset(t)dt$$

Implies

$$\int_0^{\|z_1-z_2\|^2} \emptyset(t)dt \le (\gamma+\delta)\int_0^{\|z_1-z_2\|^2} \emptyset(t)dt$$

Which is a contradiction as $\alpha+\beta+2\eta+\gamma+\delta<1$.

Thus occurs the uniqueness.

Theorem 3.3:
Let S and T be continuous mapping of Hilbert space (X,d) into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping A of X into SX$\cap TX$, which commute with S and T and satisfies the inequality

$$\int_0^{\|Ax-Ay\|^{p/q}} \emptyset(t)dt \le \alpha\int_0^{\|Ax-Sx\|^{p/q}} \emptyset(t)dt + \beta\int_0^{\|Ay-Ty\|^{p/q}} \emptyset(t)dt + \gamma\int_0^{\|Ax-Sy\|^{p/q}} \emptyset(t)dt + \delta\int_0^{\|Ax-Ty\|^{p/q}} \emptyset(t)dt + \eta\int_0^{\max\{\|Ax-Tx\|^{p/q},\|Ay-Sy\|^{p/q}\}} \emptyset(t)dt$$

For all x,y in X ; where $\alpha, \beta, \gamma, \delta, \eta$ are non-negative reals with $0<\alpha+\beta+\eta+2(\gamma+\delta)2^{p/q}<1$ where p,q are positive real numbers with $p/q<1$ Also $\emptyset$:[0,+∞) $\rightarrow$[0,+∞) is a Lebesgue integrable mapping which is summable on each

compact subset of [0,+∞) , non negative ,and such that for each $\varepsilon>0$, $\int_0^{\varepsilon}\emptyset(t)dt$, then S,T and A then have a unique common fixed point .

Proof : First of all we prove that the existence of such a mapping A is necessary. For this suppose ,Sz=z=Tz for some z in X.

Let us define a mapping A of X into X by Ax=z for all x in x. Then clearly , A is continuous mapping of X into SX$\cap TX$. Since, Sx, Tx $\in X$, for all

x $\in X$ and Ax=z for all x$\in X$, we get
ASx = z ,SAx =Sz =z ,ATx=z,TAx=Tz=z.

Hence, A commutes with S and T. Now for any $\alpha, \beta, \gamma, \delta, \eta$ with

$0<\alpha+\beta+\eta+2(\gamma+\delta)2^{p/q}<1$, it is observe that

$$\int_0^{\|Ax-Ay\|^{p/q}} \emptyset(t)dt \le \alpha\int_0^{\|Ax-Sx\|^{p/q}} \emptyset(t)dt + \beta\int_0^{\|Ay-Ty\|^{p/q}} \emptyset(t)dt + \gamma\int_0^{\|Ax-Sy\|^{p/q}} \emptyset(t)dt + \delta\int_0^{\|Ax-Ty\|^{p/q}} \emptyset(t)dt + \eta\int_0^{\max\{\|Ax-Tx\|^{p/q},\|Ay-Sy\|^{p/q}\}} \emptyset(t)dt$$

Which gives $0\le\alpha\int_0^{\|Ax-Sx\|^{p/q}} \emptyset(t)dt + \beta\int_0^{\|Ay-Ty\|^{p/q}} \emptyset(t)dt$

$$+ \gamma \int_0^{\|Ax-Sy\|^{p/q}} \emptyset(t)dt + \delta \int_0^{\|Ax-Ty\|^{p/q}} \emptyset(t)dt$$
$$+ \eta \int_0^{\max\{\|Ax-Tx\|^{p/q}, \|Ay-Sy\|^{p/q}\}} \emptyset(t)dt$$

For all x,y in X. This proves the existence of such a mapping A is necessary.

To prove sufficient part, sequence $\{x_n\}$ is constructed as follows. Let $x_0 \in X$ be an arbitrary point .

Since AX $\subset$ SX , we choose a point x_1 in X such that S $x_1 = A\, x_0$.

Also AX $\subset$ TX and hence we can choose $x_2 \in X$ such that T$x_2 = A\, x_1$.

Continuing in this way, we get a sequence $\{x_n\}$ as follows :

S $x_{2n-1} = A\, x_{2n-2}$, T$x_{2n} = A\, x_{2n-1}$ n = 1,2,3 . ………

We proceed to show that $\{Ax_n\}$ is a Cauchy sequence .

For this we have the inequality,

$$\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt \le \alpha \int_0^{\|Ax_{2n+1}-Sx_{2n+1}\|^{p/q}} \emptyset(t)dt$$
$$+ \beta \int_0^{\|Ax_{2n}-Tx_{2n}\|^{p/q}} \emptyset(t)dt$$
$$+\gamma \int_0^{\|Ax_{2n+1}-Sx_{2n}\|^{p/q}} \emptyset(t)dt + \delta \int_0^{\|Ax_{2n+1}-Tx_{2n}\|^{p/q}} \emptyset(t)dt$$
$$+ \eta \int_0^{\max\{\|Ax_{2n+1}-Tx_{2n+1}\|^{p/q}, \|Ax_{2n}-Sx_{2n}\|^{p/q}\}} \emptyset(t)dt$$

$$= \alpha \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt + \beta \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt$$
$$+ \gamma \int_0^{\|Ax_{2n+1}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt + \delta \int_0^{\|Ax_{2n+1}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt$$
$$+ \eta \int_0^{\max\{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}, \|Ax_{2n}-Ax_{2n-1}\|^{p/q}\}} \emptyset(t)dt$$

$$= \alpha \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt + \beta \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt$$
$$+(\gamma + \delta) \int_0^{\|Ax_{2n+1}-Ax_{2n}+Ax_{2n}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt$$
$$+ \eta \int_0^{\max\{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}, \|Ax_{2n}-Ax_{2n-1}\|^{p/q}\}} \emptyset(t)dt$$

$$\leq \alpha \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt + \beta \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt$$

$$+(\gamma+\delta)2^{p/q}\left[\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt + \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt\right]$$

$$+\eta \int_0^{\max\left\{\|Ax_{2n+1}-Ax_{2n}\|^{p/q},\|Ax_{2n}-Ax_{2n-1}\|^{p/q}\right\}} \emptyset(t)dt$$

Because $\|a+b\|^p \leq 2^p(\|a\|^p+\|a\|^p)$ for $0< p<1$

---3.3.1

Case : 1 If

$$\max\left\{\|Ax_{2n+1}-Ax_{2n}\|^{p/q},\|Ax_{2n}-Ax_{2n-1}\|^{p/q}\right\}=\|Ax_{2n+1}-Ax_{2n}\|^{p/q}$$

From (1.5.1)

$$\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt \leq \alpha \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt$$

$$+\beta \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt$$

$$+(\gamma+\delta)2^{p/q}\left[\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt + \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt\right]$$

$$+\eta \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt$$

$$=\left(\alpha+\eta+(\gamma+\delta)2^{p/q}\right)\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt$$

$$+\left(\beta+(\gamma+\delta)2^{p/q}\right)\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt$$

Which leads to

$$\left(1-\alpha-\eta-(\gamma+\delta)2^{p/q}\right)\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt \leq$$

$$\left(\beta+(\gamma+\delta)2^{p/q}\right)\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt$$

Which yields

$$\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt \leq \left(\frac{\beta+(\gamma+\delta)2^{p/q}}{1-\alpha-\eta-(\gamma+\delta)2^{p/q}}\right)\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt$$

(3.3.2)

Similarly we have

$$\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt \le \left(\frac{\beta+(\gamma+\delta)2^{p/q}}{1-\alpha-\eta-(\gamma+\delta)2^{p/q}}\right) \int_0^{\|Ax_{2n-1}-Ax_{2n-2}\|^{p/q}} \emptyset(t)dt$$

(3.3.3)

Hence , one concludes that

$$\int_0^{\|Ax_{n+1}-Ax_{n}\|^{p/q}} \emptyset(t)dt \le \left(\frac{\beta+(\gamma+\delta)2^{p/q}}{1-\alpha-\eta-(\gamma+\delta)2^{p/q}}\right) \int_0^{\|Ax_{n}-Ax_{n-1}\|^{p/q}} \emptyset(t)dt$$

Which implies that

$$\int_0^{\|Ax_{n+1}-Ax_{n}\|} \emptyset(t)dt \le \left(\frac{\beta+(\gamma+\delta)2^{p/q}}{1-\alpha-\eta-(\gamma+\delta)2^{p/q}}\right)^{q/p} \int_0^{\|Ax_{n}-Ax_{n-1}\|} \emptyset(t)dt$$

$$\le \lambda_1 \int_0^{\|Ax_{n}-Ax_{n-1}\|} \emptyset(t)dt$$

Where $\lambda_1 = \left(\frac{\beta+(\gamma+\delta)2^{p/q}}{1-\alpha-\eta-(\gamma+\delta)2^{p/q}}\right)^{q/p}$

Case : 2 If

$$\max\left\{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}, \|Ax_{2n}-Ax_{2n-1}\|^{p/q}\right\} = \|Ax_{2n}-Ax_{2n-1}\|^{p/q}$$

From (3.3.1)

$$\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt \le \alpha \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt$$

$$+ \beta \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt$$

$$+(\gamma+\delta)2^{p/q} \left[\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt + \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt\right]$$

$$+\eta \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt$$

$$=\left(\alpha+(\gamma+\delta)2^{p/q}\right) \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt$$

$$+\left(\beta+\eta+(\gamma+\delta)2^{p/q}\right) \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \emptyset(t)dt$$

This leads to

$$\left(1-\alpha-(\gamma+\delta)2^{p/q}\right) \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \emptyset(t)dt \le$$

$$\left(\beta+\eta+(\gamma+\delta)2^{p/q}\right)\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}}\phi(t)dt$$

Which yields

$$\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}}\phi(t)dt \le \left(\frac{\beta+\eta+(\gamma+\delta)2^{p/q}}{1-\alpha-(\gamma+\delta)2^{p/q}}\right)\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}}\phi(t)dt$$

(3.3.4)

Similarly we have

$$\int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}}\phi(t)dt \le \left(\frac{\beta+\eta+(\gamma+\delta)2^{p/q}}{1-\alpha-(\gamma+\delta)2^{p/q}}\right)\int_0^{\|Ax_{2n-1}-Ax_{2n-2}\|^{p/q}}\phi(t)dt \quad (3.3.5)$$

Hence , one concludes that

$$\int_0^{\|Ax_{n+1}-Ax_{n}\|^{p/q}}\phi(t)dt \le \left(\frac{\beta+\eta+(\gamma+\delta)2^{p/q}}{1-\alpha-(\gamma+\delta)2^{p/q}}\right)\int_0^{\|Ax_{n}-Ax_{n-1}\|^{p/q}}\phi(t)dt$$

Which implies that

$$\int_0^{\|Ax_{n+1}-Ax_{n}\|}\phi(t)dt \le \left(\frac{\beta+\eta+(\gamma+\delta)2^{p/q}}{1-\alpha-(\gamma+\delta)2^{p/q}}\right)^{q/p}\int_0^{\|Ax_{n}-Ax_{n-1}\|}\phi(t)dt$$

$$\le \lambda_2\int_0^{\|Ax_{n}-Ax_{n-1}\|}\phi(t)dt$$

Where $\lambda_2 = \left(\frac{\beta+\eta+(\gamma+\delta)2^{p/q}}{1-\alpha-(\gamma+\delta)2^{p/q}}\right)^{q/p}$

From both cases

Taking $\lambda = \max\{\lambda_1, \lambda_2\}$

$0 < \lambda < 1$ as we have

$$0 < \alpha+\beta+\eta+2(\gamma+\delta)2^{p/q} < 1$$

Now ,from the both cases

$$\int_0^{\|Ax_{n+1}-Ax_{n}\|}\phi(t)dt \le \lambda\int_0^{\|Ax_{n}-Ax_{n-1}\|}\phi(t)dt$$

$$\le \lambda^2\int_0^{\|Ax_{n-1}-Ax_{n-2}\|}\phi(t)dt$$

………….. …………………

$$\leq \lambda^n \int_0^{\|Ax_1 - Ax_0\|} \emptyset(t)dt$$

For large n. Now , it can be seen that $\{Ax_n\}$ is a Cauchy sequence and so it has a limit x in X. Since sequences $\{Sx_{2n+1}\}$ and $\{Tx_{2n}\}$ are subsequences of $\{Ax_n\}$, they have the same limit z. As S and A are commuting mapping, we can have

$$Sz = \lim_{n\to\infty} S\,Ax_{2n+1}$$
$$= \lim_{n\to\infty} A\,S\,x_{2n+1}$$
$$= Az$$

Similarly ,we get Tz = Az

This gives us ,

Tz = Az = Sz

It comes out that

$$\int_0^{\|Az-AAz\|^{p/q}} \emptyset(t)dt \leq \alpha \int_0^{\|Az-Sz\|^{p/q}} \emptyset(t)dt + \beta \int_0^{\|AAz-TAz\|^{p/q}} \emptyset(t)dt$$
$$+ \gamma \int_0^{\|Az-SAz\|^{p/q}} \emptyset(t)dt + \delta \int_0^{\|Az-TAz\|^{p/q}} \emptyset(t)dt$$
$$+ \eta \int_0^{\max\left\{\|Az-Tz\|^{p/q},\|AAz-SAz\|^{p/q}\right\}} \emptyset(t)dt$$

Commutatively of A with S and T gives the following by

$$\|AAz - TAz\| = \|AAz - ATz\| = \|AAz - AAz\| = 0$$

And

$$\|Az - SAz\| = \|Az - AAz\| \text{ and } \|Az - TAz\| = \|Az - AAz\|$$

Also,

$$\|AAz - SAz\| = \|AAz - ASz\| = \|AAz - AAz\| = 0$$

Resorting to these, we arrive at

$$\int_0^{\|Az-AAz\|^{p/q}} \emptyset(t)dt \leq (\gamma + \delta) \int_0^{\|Az-AAz\|^{p/q}} \emptyset(t)dt$$

Since $\alpha + \beta + \eta + 2(\gamma + \delta)2^{p/q} < 1$, we must have Az = AAz

Finally, putting Az = z_1 , we have Az_1 = Az = AAz = z_1

Similarly, Tz_1 = TAz = ATz =AAz =Az = z_1,

Sz_1 = z_1 .

So, z_1 is a fixed point of S ,T and A.

Next , to show uniqueness of this Common fixed point , let us suppose that z_2 ia also a common fixed point of S ,T and A other than z_1 .

Then Sz_2 = z_2 , Tz_2 = z_2 , Az_2 = z_2 , and also $\|z_1 - z_2\| \neq 0$.

Hence, it follows that

$$\int_0^{\|z_1-z_2\|^{p/q}} \emptyset(t)dt = \int_0^{\|Az_1-Az_2\|^{p/q}} \emptyset(t)dt$$
$$\leq \alpha \int_0^{\|Az_1-Sz_1\|^{p/q}} \emptyset(t)dt + \beta \int_0^{\|Az_2-Tz_2\|^{p/q}} \emptyset(t)dt$$

$$+ \gamma \int_0^{\|Az_1 - Sz_2\|^{p/q}} \emptyset(t)dt + \delta \int_0^{\|Az_1 - Tz_2\|^{p/q}} \emptyset(t)dt$$

$$+ \eta \int_0^{\max\{\|Az_1 - Tz_1\|^{p/q}, \|Az_2 - Sz_2\|^{p/q}\}} \emptyset(t)dt$$

Implies

$$\int_0^{\|z_1 - z_2\|^{p/q}} \emptyset(t)dt \leq (\gamma + \delta) \int_0^{\|z_1 - z_2\|^{p/q}} \emptyset(t)dt$$

Which is a contradiction as $\alpha + \beta + \eta + 2(\gamma + \delta)2^{p/q} < 1$.

Thus occurs the uniqueness.

References:

[1] A. Branciari, A fixed point theorem for mappings satisfying a general con-tractive condition of integral type, Int.J.Math.Math.Sci, 29(2002), no.9, 531 - 536.
[2] B.E Rhoades, Two fixed point theorems for mappings satisfying a general contractive condition of integral Type, International Journal of Mathematics and Mathematical Sciences, 63, (2003), 4007 - 4013.
[3]F.Brain, “Mapping with a common,” Math. Sem. Notes, vol.7, (1979)pp.81- 83,
[4] G.Abdulvalieva and D.S.K.Verbovetskyi, ”Fixed point theorems for non-commutative functions ,” J.Math.Anal.Appl.401 , (2013)pp. 436-446,
[5] H.L.Royden,Real Analysis ,PHI ,NY,ch.6, (1988)pp.122-123,.
[6] J. Gerald, “Commuting mappings and fixed points, Amer.Math” vol.83, (1976) pp.261-263
[7] Krishna P.Patel and G.M.Deheri “Extension of Some Common Fixed Point Theorems “ Vol.3(2013)pp.329-335.
[8] M. Moosaei,” Fixed point theorems in convex metric spaces,”Fixed Point Theory Appl. (2012) pp.164.
[9] P.V.Koparde and B.B. Waghmode , “ On common fixed point theorem for three mappings in Hilbert space ,” The Math .Ed., vol.28, no.1 (1994),pp. 6-9 [10]R.K.Bisht and R.U. Joshi,” Common fixed point theorems of weakly reciprocal continuous maps,” Journal of the Indian Math.Soc.vol. 79, (1968) (2012)no.1-4, ,.
[11] R. Kannan, Some results on fixed points, Bull.Calcutta Math. Soc. 60(1968), 71-76
[12] R.P.Pant, R.K.Bisht, and D.Arora,” Weak reciprocal continuity and fixed point theorems ,” Ann Univ. Ferrara,vol.57, (2011)pp. 181-190,.
[13] S. Banach, Sur les oprations dans les ensembles abstraits et leur application aux quations intgrals, Fund. Math.3,(1922)133181 (French).
[14] S.S. Yao and L.H.Qiu ,” Hybrid iteration method for fixed points of nonself nonexpansive mapping in real Banach space and its applications,” Int .Math., forum 7,no.6 , (2012)pp. 251-258,.
[15] W. Takahashi, “A convexity in metric spaces and nonexpansive mapping- 1,” Kodai Math.Sem.Rep., vol.22,(1970) pp. 142-149,.