

Fixed Point Theorems in 2- Metric Space with Continuous Convex Structure

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Abstract
In the present paper fixed point theorems are proved for 2- metric spaces with continous convex structure for more generalized conditions.
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1. Introduction & Preliminaries: Since Banach's fixed point theorem in 1922, because of its simplicity and usefulness, it has become a very popular tool in solving the existence problems in many branches of nonlinear analysis. For some more results of the generalization of this principle.

Theorem 1A: Banach [1] The well known Banach contraction principle states that "If X is complete metric space and T is a contraction mapping on X into itself, then T has unique fixed point in X".

Theorem 1 B: Kanan [16] proved that "If T is self mapping of a complete metric space X into itself satisfying:

$$d(Tx,Ty) \leq \eta\,[d(x,Tx) + d(y,Ty)]$$

$for\ all\ \ x,y \in X,\ and\ \ \eta \in \left[0,\frac{1}{2}\right].$ Then T has unique fixed point in X.

Theorem 1C: Fisher [9] proved the result with

$$d(Tx,Ty) \leq \ \mu\,[d(Tx,x) + d(Ty,y)] \ + \ \delta\, d(x,y)$$

$for\ all\ \ x,y \in X,\ and\ \ \mu,\delta \in \left[0,\frac{1}{2}\right].$ Then T has unique fixed point in X.

Theorem 1D: A similar conclusion was also obtained by Chaterjee [3].

$$d(Tx,Ty) \leq \mu[\,d(Ty,x) + d(Tx,y)]$$

$for\ all\ \ x,y \in X,\ and\ \ \eta \in \left[0,\frac{1}{2}\right].$ Then T has unique fixed point in X.

Theorem 1E: Ciric [5] proved the result

$$d(Tx,Ty) \leq \ \eta[\,d(x,Tx) + d(y,Ty)] + \mu[d(x,Ty) + d(y,Tx)]$$

$+\delta\, d(x,y)$

$for\ all\ \ x,y \in X,\ and\ \ \eta,\mu,\delta \in [\,0,1).$ Then T has unique fixed point in X.

Theorem 1F: Reich [22] proved the result

$$d(Tx,Ty) \leq \ \ \mu[d(x,Ty) + d(y,Tx)] + \delta\, d(x,y)$$

$for\ all\ \ x,y \in X,\ and\ \ \mu,\delta \in [\,0,1).$ Then T has unique fixed point in X.

Theorem1 G: In 1977, the mathematician Jaggi [14] introduced the rational expression first

$$d(Tx,Ty) \leq \beta\,\frac{d(x,Tx)d(y,Ty)}{d(x,y)} \ + \ \ \delta d(x,y)$$

for all $x,y \in X, x \neq y\,,\ \ \beta,\delta \in [\,0,1)$ and $0 \leq \ \delta + \beta < 1.$ Then T has unique fixed point in X.

Theorem1H: In 1980 the mathematicians Jaggi and Das [15] obtained some fixed point theorems with the mapping satisfying:

$$d(Tx,Ty) \leq \alpha\, d(x,y) \ + \ \beta\,\frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(y,Tx)+d(x,Ty)}$$

for all $x, y \in X, x \neq y$, $\beta, \delta \in [0,1)$ and $0 \leq \delta + \beta < 1$. Then T has unique fixed point in X.

These are extensions of Banach contraction principle [1] in terms of a new symmetric rational expression. Takahashi [30] has introduced the definition for convexity in metric space and generalized some fixed point theorems previously proved for the Banach space. Subsequently, Mochado [28], Tallman [31], Naimpally and Singh [29], Guay and Singh [26], Hadzic and Gajic [27] were among others who obtained results in this setting. This paper is a continuation of the investigation in the same setting in form of Altering distance function motivated by Sharma and Devangan [23], Sharma , Sharma, Iskey [24]

To prove the main result we need following modified definitions:

Definition2.1. Let X be a 2-metric space and I be the closed unit interval. A mapping $W: X \times X \times I \to X$ is said to be a convex structure on X if for all $x, y \in X, \lambda \in I$, a > 0
$d(u, (W(x, y, \lambda), a) \leq \lambda d(u, x, a) + (1-\lambda) d(u, y, a)$, for all $u \in X$.
The metric space(X, d) together with a convex structure is called the Takahashi convex metric space.
Any subset of a Banach space is a Takahashi convex metric space with
$W(x, y, \lambda) = \lambda x + (1-\lambda)$.

Definition 2.2 Let X be a convex 2-metric space. A nonempty subset K of X is said to be convex if and only if $W(x, y, \lambda) \in K$ whenever $x, y \in K, \lambda \in I$.
Takahashi [5] has shown that the open and closed balls are convex and that an arbitrary intersection of convex sets is also convex.
For an arbitrary $A \subset X$, let

(1) $$\widehat{W}(A) = \{W(x, y, \lambda): x, y \in A, \lambda \in I\}.$$

It is easy to see that
$\widehat{W}: P(X) \to P(X)$ is a mapping with the properties:

(i) $A \subset \widehat{W}(A)$, for $A \subset X$,

(ii) $A \subset B \Rightarrow \widehat{W}(A) \subset \widehat{W}(B)$, for $A, B \in P(X)$,

(iii) $\widehat{W}(A \cap B) \subset W(A) \cap \widehat{W}(B)$, for any $A, B \in P(X)$.

Using this notation we can see that K is convex iff $\widehat{W}(K) \subset K$.

Definition2.3. A convex 2-metric space X will be said to have property (C) iff every bounded decreasing set of nonempty closed convex subset of X has nonempty intersection.

Definition 2.4. Let X be a convex 2-metric space and A be a nonempty closed, convex bounded set in X. For $x \in X$, a > 0 let us set
$r_x(A) = \sup_{y \in A} d(x, y, a)$,
And $r(A) = \inf_{x \in A} r(A)$.
We thus define $A_C = \{x \in A: r_x(A) = r(A)\}$ to be the centre of A.
We denote the diameter of a subset A of X by
$\delta(A) = \sup\{d(x, y, a): x, y \in A\}$.

Definition2. 5. A point $x \in A$ is a diametral point of A iff
$\sup_{y \in A} d(x, y, a) = \delta(A)$.

Definition2.6. A convex 2-metric space X is said to have normal structure iff for each closed bounded, convex subset A of X, containing at least two points, there exists $x \in A$, which is not a diametral point of A.

Remarks Any compact convex 2-metric space has a normal structure.

Definition 2.7. A Convex hull of the set $A (A \subset X)$ is the intersection of all convex sets in X containing A, an is denoted by convex A.
It is obvious that if A is a convex set, then
$\widehat{W}^n(A) = \widehat{W}\left(\widehat{W}(\widehat{W}(A) \ldots)\right) \subset A$ for any $n \in N$.
If we set
$A_n = \widehat{W}^n(A), (A \subset X)$,
Then the sequence $\{A_n\}_{n \in} N$ will be increasing and lim $\sup A_n$ exists, and lim $\sup A_n = \lim\inf A_n = \lim A_n = \cup_{n=1}^{\infty} A_n$.
In 1984, M.S. Khan , M. Swalech and S.Sessa [19] expanded the research of the metric fixed point theory to a

new category by introducing a control function which they called an altering distance function. Motivated by them we find the same for 2- metric spaces as follows

Definition 2.8 ([19]) A function $\psi : \Re_+ \to \Re_+$ is called an altering distance function if the following properties are satisfied:

(ψ_1) $\psi(t) = 0 \Leftrightarrow t = 0$

(ψ_2) ψ is monotonically non-decreasing.

(ψ_3) ψ is continuous.

By ψ we denote the set of the all altering distance functions.

Theorem2.9 ([49]) Let (M,d) be a complete 2-metric space, let $\psi \in \Psi$ and let $S : M \to M$ be a mapping a > 0 which satisfies the following inequality

$$\Psi[d(Sx,Sy,a)] \le \alpha \Psi[d(x,y,a)]$$

For all $x, y \in M$ and for some $0 < a < 1$. Then S has a unique fixed point $z_0 \in M$ and moreover for each $x \in M$ $\lim_{n\to\infty} S^n x = z_0$

Lemma 2.10 Let (M,d) be 2- metric space. Let $\{x_n\}$ be a sequence in M such that

$$\lim_{n\to\infty} \Psi[d(x_n, x_{n+1}, a)] = 0$$

If $\{x_n\}$ is not a Cauchy sequence in M , then there exist an $\varepsilon_0 > 0$ and sequences of integers positive $\{m(k)\}$ and $\{n(k)\}$ with

$$m(k) > n(k) > k$$

Such that $\Psi\left[d(x_{m(k)}, x_{n(k)}, a)\right] \ge \epsilon_0$, $\Psi\left[d(x_{m(k-1)}, x_{n(k)}, a)\right] < \epsilon_0$

(i) $\lim_{k\to\infty} \Psi\left[d(x_{m(k-1)}, x_{n(k+1)}, a)\right] = \epsilon_0$

(ii) $\lim_{k\to\infty} \Psi\left[d(x_{m(k)}, x_{n(k)}, a)\right] = \epsilon_0$

(iii) $\lim_{k\to\infty} \Psi\left[d(x_{m(k-1)}, x_{n(k)}, a)\right] = \epsilon_0$

Remark 2.11 It is easy to get

$$\lim_{k\to\infty} \Psi\left[d(x_{m(k+1)}, x_{n(k+1)}, a)\right] = \epsilon_0$$

Definition (2.12) A 2- metric space is a space X in which for each triple of points x, y, z, there exists a real function d (x,y,z,) such that

$[M_1]$ to each pair of distinct points x,y,z,

d (x,y,z) ≠ 0

$[M_2]$ d (x,y,z) = 0 when at lest two of x,y,z are equal

$[M_3]$ d (x,y,z) = d (y,z,x) = d (x,z,y)

$[M_4]$ d (x,y,z) ≤ d (x,y,v) + d (x,v,z) + d (v,y,z) for all x,y,z, v in X.

Definition (2.13): A sequence $\{x_n\}$ in a 2-metic space (X,d) is said to be convergent at x if

$$\lim_{n\to\infty} d(x_n, x, z) = 0 \text{ for all z in X.}$$

Definition (2.14) A sequence $\{x_n\}$ in a 2-metric space, (x, d) is said to be Cauchy sequence if

$$\lim_{m,n\to\infty} d(x_n, x, z) = 0 \text{ for all z in X.}$$

Definition (2.15) A 2-metic space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

Also, we need the following propositions:

Proposition 1[23]. Let X be a convex 2- metric space. Then

(2) $conv\ A = \lim A_n = \bigcup_{n=1}^{\infty} A_n$, $(A \subset X)$

In the remaining part of this paper (X,d) will denote a convex 2-metric space.
Proposition 2 [23]. For any subset A of (X,d)
$\delta(conv\ A) = \delta(A)$.

3. Main result

Now we prove the following

Theorem 3.1. Let a function $\psi : \Re_+ \to \Re_+$ is an altering distance function. (X,d) be 2- metric space with continuous convex structure and let K be a closed convex bounded subset of (X,d) with normal structure and property(C)
If $A{:}K \to K$ is a continuous mapping such that for $x, y \in K$, a > 0

(3) $\Psi d(Ax,Ay,a) \le \Psi \max \begin{Bmatrix} d(x,y,a), d(x,Ax,a), d(y,Ay,a), d(x,Ay,a), d(y,Ax,a) \\ d(x,A^2x,a), d(y,A^2y,a), d(Ax,A^2x,a), d(Ay,A^2y,a) \end{Bmatrix}$

Then A has a fixed point.
Proof. Let F be a family of non-empty closed convex subsets $F \subset K$ so that$A(F) \subset F$, then F is non-empty since$K \in F$. We partially order F by inclusion, and let $S = \{F_i\}_{i\in\Delta}$ be the decreasing chain in F. Then by Property (C) we have that
$F_0 = \bigcap_{i\in I} F_i \neq \emptyset.$
So,
$F_0 \in F$.
Therefore, any chain in F has a greatest lower bound, and by Zorn's Lemma there is a minimal member $\mathcal{F}$ in F. We claim that F is a singleton set. If not, then, as shown by Takahashi [5], the centre of D, denoted by F_C, is a non-empty proper closed convex subset of F. Now, it is easy to see that
$\delta(F_C) \le r(F) \le \delta(F)$.
Now, let us define a sequence $F_0 = F_c$ and
$F_{k+1} = conv\big(F_k \cup A(F_k)\big), k = 0,1,\dots$
Clearly, $F_k \subset F_{k+1}, (K = 0,1,\dots.)$. Thus we shall prove by induction that

(4) $\delta_k = \delta(F_k) \le r(F) = r$, for any $k \in N$.

For $k = 0$ (5) is valid. Suppose that it is valid for $k = 0,1,\dots,m,$ then we show that it is also valid for $k = m+1$.
By definition of $\delta(F)$ for any sequence $\{\varepsilon_n\}, \varepsilon_n > 0 (n \in N), \lim_{n\to\infty} \varepsilon_n = 0,$ there exist $\tilde{x}_n, \tilde{y}_n \in F_{m+1}$, so that
$\delta_{m+1} - \varepsilon_n \le d(\tilde{x}_n, \tilde{y}_n)$.
Then, by proposition 2 we have three cases:

(i) $\tilde{x}_n, \tilde{y}_n \in F_m (n = 1,2,\dots)$

(ii) $\tilde{x}_n, = x_n\ \tilde{y}_n = Ay_n (x_n, y_n \in F_m, n = 0,1,\dots)$

(iii) $\tilde{x}_n = Ax_n\ \ \tilde{y}_n = Ay_n (x_n, y_n \in F_m, n = 0,1,\dots)$

Considering the first case it is clear that $\delta_{m+1} \le r$. So, let us see the second one. For any $x \in F_0$ thus we have

(5) $d(x,Ax,a) \le r$

We assume that (6) is valid for $x \in F_k (k = 0,1,\dots.,m-1)$ and prove that it is valid for $k = m$.
For any $x \in F_m$, by preposition 1, $x \in \widehat{W}^{n_0}\big(F_{m-1} \cup A(F_{m-1})\big)$ for some $n_0 \in N$. Then

(6) $\Psi d(x,Ax,a) \le \sum_{j\in I_1} \gamma j \Psi d\big(x_j, Ax, a\big) + \sum_{j\in I_2} \gamma j \Psi d\big(Ax_j, Ax, a\big),$

For $x_j \in F_{m-1}, j \in I = I_1 \cup I_2$, (I-finiter set), $I_1 \cap I_2 = \emptyset$ and $\sum_{j\in I} \gamma j = 1, \gamma_j \ge 0$ for $j \in I$. In (7) is sufficient to look only for the case wthen $\sum_{j\in I} \gamma j \neq 0$.
Further, we have

$\Psi d(x,Ax,a) \le \sum_{j\in I_1} \gamma j \Psi d\big(x_j, Ax, a\big) + \sum_{j\in I_2^{(1)}} \gamma_j \Psi d\big(Ax_j, x, a\big)$

$\sum_{j\in I_2^{(2)}} \gamma_j \Psi d\big(x_j, Ax_j, a\big) + \sum_{j\in I_2^{(3)}} \gamma_j \Psi d\big(x_j, Ax, a\big)$

$\sum_{j\in I_2^{(4)}} \gamma_j \Psi d\big(x_j, Ax, a\big) + \sum_{j\in I_2^{(5)}} \gamma_j \Psi d\big(x_j, Ax_j, a\big)$

$\sum_{j\in I_2^{(6)}} \gamma_j \Psi d(x_j, A^2 x_j, a) + \sum_{j\in I_2^{(7)}} \gamma_j \Psi d(x, A^2 x, a)$

$\sum_{j\in I_2^{(8)}} \gamma_j \Psi d(Ax_j, A^2 x_j, a) + \sum_{j\in I_2^{(9)}} \gamma_j \Psi d(Ax, A^2 x, a)$

Where we suppose

for $I \in I_2^{(1)}$ that $\Psi\ d(Ax_j, Ax, a) \leq \Psi d(x_j, x, a)$

for $I \in I_2^{(2)}$ that $\Psi\ d(Ax_j, Ax, a) \leq \Psi d(x_j, Ax_j, a)$

for $I \in I_2^{(3)}$ that $\Psi\ d(Ax_j, Ax, a) \leq \Psi d(x, Ax, a)$

for $I \in I_2^{(4)}$ that $\Psi\ d(Ax_j, Ax, a) \leq \Psi d(x_j, Ax, a)$

for $I \in I_2^{(5)}$ that $\Psi d(Ax_j, Ax, a) \leq \Psi d(x, Ax_j, a)$

for $I \in I_2^{(6)}$ that $\Psi d(Ax_j, Ax, a) \leq \Psi d(x_j, A^2 x_j, a)$

for $I \in I_2^{(7)}$ that $\Psi\ d(Ax_j, Ax, a) \leq \Psi d(x, A^2 x, a)$

for $I \in I_2^{(8)}$ that $\Psi\ d(Ax_j, Ax) \leq \Psi d(Ax, A^2 x_j)$

for $I \in I_2^{(9)}$ that $\Psi d(Ax_j, Ax, a) \leq \Psi d(Ax, A^2 x, a)$.

Now, using the hypothesis, one can see that

$\Psi d(x, Ax, a) \leq \sum_{j\in I_1} \gamma_j \Psi d(x_j, Ax, a) + r \sum_{j\in I_2^{(1)}} \gamma_j$

$+ \sum_{j\in I_2^{(2)}} \gamma_j \Psi d(x_j, Ax_j, a) + \sum_{j\in I_2^{(3)}} \gamma_j \Psi d(x_j, Ax, a)$

$+ \sum_{j\in I_2^{(4)}} \gamma_j \Psi d(x_j, Ax, a) + \sum_{j\in I_2^{(5)}} \gamma_j \Psi d(x_j, Ax_j, a)$

$+ \sum_{j\in I_2^{(6)}} \gamma_j \Psi d(x_j, A^2 x_j, a) + \sum_{j\in I_2^{(7)}} \gamma_j \Psi d(x_j, A^2 x, a)$

$+ \sum_{j\in I_2^{(8)}} \gamma_j \Psi d(Ax_j, A^2 x_j, a) + \sum_{j\in I_2^{(9)}} \gamma_j \Psi d(Ax_j, A^2 x, a)$

Since by induction, similarly, we have

$\Psi d(x, Ax, a) \leq \sum_{k\in j_j^{(1)}} \beta_k \Psi d(\hat{x}k, x_j, a)$

$+ \sum_{k\in j_j^{(2)}} \beta_k \Psi d(\hat{x}k, A\hat{x}k, a) + \sum_{k\in j_j^{(3)}} \beta_k \Psi d(x_j, Ax_j, a)$

$+ \sum_{k\in j_j^{(4)}} \beta_k \Psi d(\hat{x}k, Ax_j, a) + \sum_{k\in j_j^{(5)}} \beta_k \Psi d(x_j, A\hat{x}k, a)$

$+ \sum_{k\in j_j^{(6)}} \beta_k \Psi d(\hat{x}k, A^2\hat{x}k, a) + \sum_{k\in j_j^{(7)}} \beta_k \Psi d(x_j, A^2 x_j, a)$

$+ \sum_{k\in j_j^{(8)}} \beta_k \Psi d(A\hat{x}k, A^2\hat{x}k, a) + \sum_{k\in j_j^{(9)}} \beta_k \Psi d(Ax_j, A^2 x, a)$,

for $\hat{x}k \in F_{m-1}(k \in J_i = \cup_{i=1}^{9} J_i^{(i)}, \sum_{k\in J_j} \beta_k = 1$ and $B_k \geq 0, k \in J_j, \sum_{k\in J_j^{(1)}} \beta_k \neq 0)$. Therefore

$d(x, Ax_j, a) \leq r$

and

$\Psi d(x, Ax, a) \leq \sum_{j\in I_1} \gamma_j \Psi d(x_j, Ax, a) + r(\sum_{j\in I_2^{(1)}} + \sum_{j\in I_2^{(2)}} + \sum_{j\in I_2^{(5)}})\gamma_j$

$+ \sum_{j\in I_2^{(3)}} \gamma_j \Psi d(x, Ax, a) + \sum_{j\in I_2^{(4)}} \gamma_j \Psi d(x, Ax, a)$

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$+ \sum_{j\in I_2^{(6)}} \gamma_j \Psi d(x_j, A^2 x_j, a) + \sum_{j\in I_2^{(7)}} \gamma_j \Psi d(x, Ax, a)$

$+ \sum_{j\in I_2^{(8)}} \gamma_j \Psi d(Ax_j, A^2 x_j, a) + \sum_{j\in I_2^{(9)}} \gamma_j \Psi d(Ax_j, A^2 x, a)$.

After not more than n_0 steps we shall that

$\Psi d(x, Ax, a) \leq \sum_{j\in I^*} \gamma_j^* \Psi d(v_j, Ax, a) + \gamma_0^* r$,

for

$\gamma_j^* \geq 0, i \in \{0\} \cup I^*$

$\gamma_0^* + \sum_{j\in I^*} \gamma_j^* = 1$

And

$v_j \in F_{0,j} \in I^*$.

Since F_0 is the centre we have that

$d(v_j, Ax, a) \leq r$,

Which implies that

$\Psi d(x, Ax) \leq r$ for all $x \in F_m$.

Similarly, we can prove that

$\Psi d(x, Ay, a) \leq r$ for all $x, y \in F_m$.

So, in the second case we have

$\delta_{m+1} - \varepsilon_n \leq \Psi d(\tilde{x}_n, \tilde{y}_n, a)$
$= \Psi d(x_n, Ay_n \leq r$,for $n \in N$,
And consequently
$\delta_{m+1} \leq r$.
Using (4) it is easy to prove this inequality for case (iii). Thus,
$\delta_m \leq r$ for all $m \in N$.
Let us define $F^{\infty} = \bigcup_{k=0}^{\infty} F_k$.
F_0 is non-empty. So, F^{∞} is non-empty too.
Since $\delta(F^{\infty}) < r\delta(F)$, F^{∞} is a closed proper subset of F.
Moreover, W is continuous and that closure of convex set is convex.
Since mapping A is continuous so,
$A(F^{\infty}) \subset F^{\infty}$ And therefore F^{∞} is a subset of F, which is a contradiction to the minimality of F. Hence, F consists of a single element which is a fixed point for A.

References

1. Banach, S. "Surles operation dans les ensembles abstraits et leur application aux equations integrals" Fund. Math. 3(1922) 133-181.
2. Bhardwaj, R.K., Rajput, S.S. and Yadava, R.N. "Application of fixed point theory in metric spaces" Thai Journal of Mathematics 5 (2007) 253-259.
3. Chatterjee, S.K. "Fixed point theorems compactes" Rend. Acad. Bulgare Sci, 25 (1972) 727-730.
4. Choudhary, S. Wadhwa, K. and Bhardwaj R. K. "A fixed point theorem for continuous function" Vijnana Parishad Anushandhan Patrika.(2007)110-113.
5. Ciric, L. B. "A generalization of Banach contraction Principle" Proc. Amer. Math. Soc. 25 (1974) 267-273.
6. Chu, S.C.and Diag, J.B. "Remarks on generalization on Banach principle of contractive mapping" J.Math.Arab.Appli.11 (1965) 440-446.
7. Das, B.K. and Gupta, S. "An extension of Banach contraction principle through rational expression" Indian Journal of Pure and Applied Math.6 (1975) 1455-1458.
8. Dubey, R.P. and Pathak, H.K "Common fixed pints of mappings satisfying rational inequalities" Pure and Applied Mathematika Sciences 31 (1990)155-161.
9. Fisher B. "A fixed point theorem for compact metric space" Publ.Inst.Math.25 (1976) 193-194.
10. Goebel, K. "An elementary proof of the fixed point theorem of Browder and Kirk" Michigan Math. J. 16(1969) 381-383.
11. Iseki, K., Sharma, P.L. and Rajput S.S. "An extension of Banach contraction principle through rational expression" Mathematics seminar notes Kobe University 10(1982) 677-679.
12. Imdad, M. and Khan T.I. "On common fixed points of pair wise coincidently commuting non-continuous mappings satisfying a rational inequality" Bull. Ca. Math. Soc. 93 (2001) 263-268.
13. Imdad, M and Khan, Q.H "A common fixed point theorem for six mappings satisfying a rational inequality" Indian J. of Mathematics 44 (2002) 47-57.
14. Jaggi, D.S. "Some unique fixed point theorems" I. J.P. Appl. 8(1977) 223-230.
15. Jaggi, D.S. and Das, B.K. "An extension of Banach's fixed point theorem through rational expression" Bull. Cal. Math. Soc.72 (1980) 261-264.
16. Kanan, R. "Some results on fixed point theorems" Bull. Calcutta Math. Soc, 60 (1969) 71-78.

17. Kundu, A. and Tiwary, K.S. "A common fixed point theorem for five mappings in metric spaces" Review Bull.Cal. Math. Soc.182 (2003) 93-98.
18. Liu, Z., Feng, C. and Chun, S.A. "Fixed and periodic point theorems in 2- metric spaces" Nonlinear Funct. & Appl. 4(2003) 497-505.
19. M. S. Khan, M. Swalech and S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Austral Math. Soc., 30 (1984) 1–9.
20. Nair, S. and Shriwastava, S. "Common fixed point theorem for rational inequality" Acta Cincia Indica 32 (2006) 275-278.
21. Naidu, S.V.R. "Fixed point theorems for self map on a 2-metric spaces" Pure and Applied Mathematika Sciences 12 (1995)73-77.
22. Reich, S. "Some remarks concerning contraction mapping" Canada. Math.Bull.14 (1971) 121-124.
23. Sharma. B.K. and Devangan C.L. "Fixed point theorem in convex metric spaces" Univ. u Novom Sadu Zb.Rad. Period. Mat. Fak. Ser. Mat.25,1(1985),9-18
24. Sharma. P.L., Sharma. B.K. and Iseki, K. "Contractive type mapping on 2-metric spaces" Math. Japonica 21 (1976) 67-70.
25. Singh, S.L., Kumar, A. and Hasim, A.M. "Fixed points of Contractive maps" Indian Journal of Mathematics 47 (2005) 51-58.
26. Guay, M.D., Singh, K. L., Fixed point of set valued mapping of convex metric spaces, Jnanabha 16 (1986), 13-22.
27. Hadzic, O., Gajic, Lj., Coincidence points for set valued mappings in convex metric spaces, Univ. u Novom Sadu Zb. Rad. Prirod.- Mat. Fak. Ser. Mat. 16 (1986), 11-25.
28. Machado, H. V.m A characterization of convex subsets of normed spaces, Kodai Math. Sem. Rep. 25 (1973), 307-320.
29. Naimpally, S. A., Singh, K. L., Fixed point and common fixed points in convex metric space, Preprint.
30. Takahashi, W., A convexity in metric space and non-expansive mapping 1, Kodai Math. Sem. Rep. 22 (1970) 142-149.
31. Tallman, L. A., Fixed point for condensing multifunctions in metric space with convex structure, Kodai Math. Sem. Rep. 29 (1979), 62-70.
32. Vajzovic, F., Fixed point theorems for nonlinear operators, Radovi Mat. 1(1985), 46-59 (in Russian).
33. Gajic, Lj., On convexity in convex metric spaces with applications, J. Nature Phys. Sci. 3 (1989), 39-48.