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Brownian Motion and the Black-Scholes Option Pricing Formula

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Abstract
The Brownian Motion of visible particles suspended in a fluid led to one of the first accurate determination of the mass of the visible molecules. Mathematical model of Brownian motion has numerous real world applications. For instance stock market fluctuations. The Black- Scholes model for calculating the premium of an option was introduced in 1973 in a paper published in Journal of Political Economy developed by three Economists –Fisher Black, Myron Scholes and Robert Merton and is world's most well known Option Pricing Model . In 1997 all was awarded Nobel Prize in Economics.
Keywords: Brownian Motion, Market fluctuations, Arbitrage Theorem, Random Walk, Hitting Time, Betting.

1. Introduction: In 1827 The English Botanist Robert Brown observed that the microscopic pollen grains suspended in water perform a continual swarming motion. The phenomenon was first explained by Einstein in 1905 who said the motion comes from the pollen being hit by the molecules in the surrounding water. The mathematical derivation of the Brownian motion was first done by Wiener in 1918 and in his honor it is often called Wiener Process.

2. Definition: A stochastic process [X (t), t ≥ 0] is said to be Brownian motion process if

1. X(0) = 0
2. [X(t), t ≥ 0] has stationery and independent increments.
3. For any t> 0, X (t) is normally distributed with mean 0 and variance $\sigma^2 t$. When σ=1 the process is called Standard Brownian Process.

3. **Random Walk**: Considering a walk in which each time unit is equally likely to take a unit step either to the left or to the right i.e. If x be the Markov chain with $P_{i,i+1} = \frac{1}{2} = P_{i,i-1}$, i= 0 $\pm 1, \pm 2, \dots\dots$ More precisely if each Δt time we take a step of size Δx either to the left or to the right with equal probabilities. If we let X (t) denote the position at time t then

X (t) = $\Delta x(x_1 + x_2 + \dots\dots\dots + x_{[\frac{t}{\Delta t}]})$ ....(1.1)

Where $X_i = \begin{cases} +1 \text{ if ith step of length } \Delta x \text{ is to right} \\ -1 \text{ if it is to left} \end{cases}$ [t/Δt] is the largest integer less than or equal to t/Δt and X_i are assumed independent with P [$X_i = 1$] = P[$X_i = -1$] = $\frac{1}{2}$

Further as X (t) is normal with mean 0 and variance t the probability density function of X is $f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}}$.

Then joint density function of X (t_1), X (t_2)......, X (t_n) for $t_1 < t_2 < \dots < t_n$ is

f ($x_1, x_2, \dots, x_n$) = $f_{t_1}(x_1). f_{t_2-t_1}(x_2 - x_1)$.............$f_{t_n-t_{n-1}}(x_n - x_{n-1})$

$$= \frac{\exp\left\{ \frac{-1}{2}\left[\frac{x_1^2}{t_1} + \frac{(x_2-x_1)^2}{t_2-t_1} + \dots\dots\dots + \frac{(x_n-x_{n-1})^2}{t_n-t_{n-1}} \right]\right\}}{(2\pi)^{\frac{n}{2}} \left[\{t_1 (t_2-t_1) \dots\dots (t_n-t_{n-1})\}^{\frac{1}{2}} \right]} \quad \dots\dots\dots (1.2)$$

Where X(t_1) =x_1 , X(t_2) =x_2 , ..., X(t_n) = x_n are equivalent to X(t_1) = x_1, X(t_2)- X(t_1) = $x_2 - x_1$ .... , X(t_n) - X(t_{n-1}) = $x_n - x_{n-1}$

From the equation (1.2) we can compute desired probabilities. Such as conditional distribution of X(s) given that X (t) = B where s$< t$ is

$f_{\frac{s}{t}}(x/B) = \frac{f_s^{(x)} f_{t-s}^{(B-x)}}{f_t^{(B)}} = K_1 \exp\left\{ -x^2/_{2s} - (B-x)^2/_{2(t-s)} \right\} = K_2 \exp\left\{ -x^2\left(\frac{1}{2s} + \frac{1}{2(t-s)}\right) + \frac{Bx}{t-s} \right\} =$

$K_2 \exp\left\{\frac{-t}{2s(t-s)}\left(x^2 - 2\frac{sB}{t}x\right)\right\} = K_3 exp\left\{\frac{(x - {}^{Bs}/_t)^2}{2s(t-s)/_t}\right\}$ where K_1, K2 and K_3 do not depend on x. Hence conditional distribution of X(s) given that X (t) = B is for $s < t$ is normal with mean given by $E[X(s)/\ X(t) = B] = \frac{s}{t}B$ and Variance given by Var $[X(s)/\ X(t) = B] = \frac{s}{t}(t-s)$.

3.1. Hitting Time: Let T_α denote the first hitting time the Brownian Process hits α. When $\alpha > 0$ we compute P $\{T_\alpha \leq t\}$ by considering P[X (t) $\geq \alpha$] and considering whether or not $T_\alpha \leq t$. This gives P[X(t) $\geq \alpha$]= P$\{X(t) \geq \alpha/T_\alpha \leq t\}P\{T_\alpha \leq t\}$+P$\{X(t) \geq \alpha/T_\alpha > t\}P\{T_\alpha > t\}$ ....(1.3)
Now if $T_\alpha \leq t$, then the process hits α at some point in [0,t] and by symmetry it is just above X or below α *at time* t. Then P $\{\{X(t) \geq \alpha/T_\alpha \leq t\} = \frac{1}{2}$

As second right hand side term of (1.3) is clearly equal to zero we see that P $\{T_\alpha \leq t\}= \alpha$ P[X(t)$\geq \alpha$]
$= \frac{2}{\sqrt{2\pi t}}\int_\alpha^\infty e^{-x^2/_{2t}}\,dx = \frac{2}{\sqrt{2\pi}}\int_{\alpha/\sqrt{t}}^\infty e^{-y^2/_2}\,dy$, $\alpha >$ ....(1.4)

If $\alpha < 0$ the distribution of T_α is same as of $T_{-\alpha}$ due to symmetry. Thus from (1.4) we get
P $\{T_\alpha \leq t\} = \frac{2}{\sqrt{2\pi}}\int_{|\alpha|/\sqrt{t}}^\infty e^{-y^2/_2}\,dy$ .... (1.5)
The maximum value of the process attains at [0,t] as follows:
For $\alpha > 0$, P[$Max\ X(s) \geq \alpha\ for\ 0 \leq s \leq t$] = P[$T_\alpha \leq t$] = 2P$\{X(t) \geq \alpha\}$ = $\frac{2}{\sqrt{2\pi}}\int_{\alpha/\sqrt{t}}^\infty e^{-y^2/_2}\,dy$.

Probability that Brownian Motion hit A before B where A > 0 and B >0 { As Brownian motion is the limit of symmetric Random walk }By Gambler's ruin problem Probability that the symmetric random walk goes up A before going down B when each step is equally likely of distances Δx with N $=({}^{(A+B)}/_{\Delta x},\ i = {}^{B}/_{\Delta x}$) equal to $B\Delta x/(A+B)\Delta x = B/A+B$.
Letting $\Delta x \to 0 \Longrightarrow P\ [\ up\ A\ before\ down\ B] = B/A+B$.

3.2. Definition: Brownian Motion with Drift: [$X(t), t \geq 0$] is a Brownian Motion with drift co-efficient μ and variance σ^2 if

1. X(0)=0
2. [$X(t), t \geq 0$] has stationery and independent increments.
3. X(t) is normally distributed with mean μ and variance σ^2
4.

3.3. Definition: **Geometric Brownian Motion**: If [$Y(t), t \geq 0$] is a Brownian Motion Process with drift co-efficient μ and variance parameter σ^2 , then the process [$X(t),\ t \geq 0$] defined by $X(t) = e^{Y(t)}$ is called Geometric Brownian Motion.
Art. (1): To compute Expected value of the process at time t given the history of the process upto time s .i.e. For s < t consider E [$X(t)/X(u)$, 0 $\leq$ u $\leq$ s]
E [$X(t)/X(u)$, 0 $\leq$u$\leq$ s] = E [$e^{Y(t)}/Y(u)$, 0 $\leq$ u $\leq$ s] = E [$e^{Y(s)+Y(t)-Y(s)}$/Y(u) ,0 $\leq$ u $\leq$ s]
= $e^{Y(s)}$[E {$e^{Y(t)-Y(s)}$/Y(u) ,0 $\leq$ u $\leq$ s}] = X(s) E[$e^{Y(t)-Y(s)}$]
Art. (2): Moment Generating Function of a random variable ω is given by

E [$e^{a\omega}$] =$e^{aE[\omega]+a^2 var(\omega)/2}$. Since Y (t)- Y(s) is normal with mean $\mu(t-s)$ and variance (t-s) σ^2, for a = 1 $\Rightarrow$ E [$e^{Y(t)-Y(s)}$] = $e^{\mu(t-s)+(t-s)\sigma^2/2}$

We get E[X (t)/X(u) ,0 $\leq$ u$\leq$ s] = X(s) $e^{(t-s)(\mu+\frac{\sigma^2}{2})}$ ....(1.6)

It is useful in the modeling of stock prices over time when percentage of charges are independently and identically distributed . If X_n be the price of some stock at time n , then it is reasonable to suppose X_n/X_{n-1} , n ≥ 1 are independently and identically distributed.

Let $Y_n = X_n/X_{n-1}$ and so $X_n = Y_n X_{n-1}$

Iterating we get $X_n = Y_n Y_{n-1} X_{n-2}$ = $Y_n Y_{n-1} Y_{n-2} X_{n-3}$ = ...= $Y_n Y_{n-1} Y_{n-2} \dots \dots Y_1 X_0$

Then $\log(X_n) = \sum_{i=1}^{n} \log(Y_i) + \log(X_0)$, since $\log(Y_i)$, t ≥ 1 are independently and identically distributed, [$\log(X_n)$] will be suitably normalized. So $[X_n]$ will be approximately Geometric Brownian Motion.

4. Arbitrage Theorem: Exactly one of the following statements is true:

a). There exist a probability vector P = ($p_1, p_2, \dots, p_n$) for which $\sum_{j=1}^{n} r_{ij}\, p_j$= 0 for all i= 1,2,....,m

b). There exist a betting vector X = ($X_1, X_2, \dots, X_m$) for which $\sum_{i=1}^{m} r_{ij}\, X_i > 0$ for all J=1,2,....,n

In other words if X be the outcome of the experiment, then the arbitrage theorem states that either is a probability vector P for X such that $E_p[r_i(X)] = 0$ for I = 1,2,....,n. Or else there is betting scheme that leads to a sure win.

Proof: The arbitrage theorem can be proved in several ways. Here we prove it by means of the duality of linear programming.

Consider the standard primal and dual linear programming
Problems:

Pr Primal	Du Dual
Az **Az**= b $Z \geq 0$ →min! z	yA $yA \leq$ c → max!

According to the duality theorem of the linear programming if the primal and the dual problems have feasible solutions then both problems have optimal solutions and the minimal value of the primal objective function is equal to the maximal value of the dual objective function.

Consider the following linear programming problem.

$$\sum_{i=1}^{m} x_i r_{ij} \geq x_{m+1} \qquad \text{.........................(2.1)}$$

X_{m+1} → max

According to the condition of the problem we would like to reach at least an amount X_{m+1} for all outcomes and beside we want that this amount should be maximal. This problem can be transformed to the standard dual linear programming problem and we can write the primal problem as follows:

$$\sum_{j=1}^{m} p_i = 1 \qquad \text{................(2.2)}$$

$p_j \geq 0$, for j= 1, 2, ..., 0 → min

Note that the condition of problem (2.2) is the same as in the arbitrage theorem. It can be easily observed that the problem (2.1) has feasible solution (e.g. x = 0 and X_{m+1} = 0). We distinguish two cases according that the problem (2.2) has or hasn't got any solution.

If the problem (2.2) has feasible solution (there exists a probability vector) then according to the duality theorem both problems have optimal solutions, the optimal value is zero. So X_{m+1} = 0 means that there is no sure win opportunity.

If the problem (2.2) has no feasible solution (there doesn't exist a probability vector) then according to the duality theorem there isn't any optimal solution and the objective function of problem (2.1) is not bounded from above. It means that X_{m+1} can be positive. In this case there is sure win for all outcomes, so there is arbitrage. The arbitrage theorem has been proved.

The arbitrage theorem has a weak form, which gives a connection for the sure not-lose opportunity instead of the sure win.

4.1. Weak arbitrage theorem: Exactly one of the following statements is true:

a). There exists a probability vector **p** = ($p_1, p_2, ..., p_n$), all of whose components are positive for which

$$\sum_{j=1}^{m} r_{ij} p_j = 0 \qquad \text{for all } i = 1, 2, ..., m,$$

b). there exists a betting vector x = ($x_1, x_2, ..., x_m$) for which $\sum_{j=1}^{m} x_i r_{ij} \geq 0$ for all j= 1, 2, ..., n, but for at least one index the strict inequality holds.

5. Main Result:

The Black-Scholes Option Pricing Formula: Consider first wager of observing the stock for a time s and then purchasing (or selling), one share with the intention that of selling (or purchasing) it, at time t, $0 \leq s \leq t \leq T$. The present value of the amount paid for the stock is $e^{-as}X(s)$, whereas the present value of the amount received is $e^{-at}X(t)$. Hence in order for the expected return of this wager to be 0 when P is the probability measure on X(t) , $0 \leq t \leq T$,we must have

$$E_p\,[e^{-at}X(t) / X(u),\ 0 \leq u \leq s] = e^{-as}X(s) \qquad \text{.... .... (3.1)}$$

Now consider the wager of the purchasing an option.

Suppose the option gives us the right to buy one share of the stock at time t for a price K, at time t, the worth of this option will be as follows.

$$\text{Worth of option at time t} = \begin{cases} X(t) - K\,,\ if\ X(t) \geq K \\ 0\,,\ if\ X(t) < K \end{cases}$$

At time t worth of the option is $(X(t) - K)^+$. Therefore present value of the worth of the option is $e^{-at}(X(t) - K)^+$. If c is the (time 0) cost of the option , Then in order for purchase the option to have expected (present value) return 0 we mut have

$$E_p[e^{-at}(X(t) - K)^+] = c \qquad \text{.......... (3.2)}$$

By Arbitrage theorem we can find a probability measure P on the set of outcomes of the equation (2.1). Then if c be the cost of an option to purchase one share at a time t at the fixed price K given in the equation (2.2) .Then no arbitrage is possible.

On the other hand if for a given prices c_i, i = 1,2,...., N. There is no probability measure P that satisfies (3.1). And the equality $c_i = [e^{-at_i}(X(t_i) - K)^+]$, for I = 1, 2 ,....., N. then a sure win is possible.

We will now present a Probability measure P on the outcome X (t), $0 \leq t \leq T$ that satisfies (3.1).

Suppose that X (t) = $x_0 e^{Y(t)}$, wherre $[Y(t), t \geq 0]$ is a Brownian Motion Process with drift co- efficient μ and variance parameter σ^2 . That is $[X(t), t \geq 0]$ is a Geometric Brownian Motion Process. From the equation (1.7) we have for s < t

$$E[X(t) / X(u), 0 \leq u \leq s] = X(s)\, e^{(t-s)(\mu+\frac{\sigma^2}{2})} \qquad \text{.... .... (3.3)}$$

If we chose μ and σ^2 such that $\mu + \frac{\sigma^2}{2} = \alpha$. Then equation (2.1) will be satisfied. That is by letting P be the probability measure governing the stochastic process $[X_0 e^{Y(t)}, 0 \leq t \leq T]$, where $Y(t)$ is the Brownian Motion with drift parameter μ and variance parameter σ^2 and where $\mu + \frac{\sigma^2}{2} = \alpha$ then the equation (2.1) is satisfied .

From the preceding if we price an option to purchase a share of stock at time t for a fixed price K by C =

$E_p[e^{-\alpha t}(X(t) - K)^+]$, then no arbitrage is possible . Since X (t) = $x_0 e^{\gamma(t)}$, where $\gamma(t)$ is normal with mean μt and variance parameter $\sigma^2 t$,

Then $ce^{\alpha t} = \int_{-\infty}^{\infty} (x_0 e^y - K)^+ \frac{1}{\sqrt{2\pi t\sigma^2}} e^{-(y-\mu t)^2}/2t\sigma^2 dy =$

$\int_{\log(k/x_0)}^{\infty} (x_0 e^y - K) \frac{1}{\sqrt{2\pi t\sigma^2}} e^{-(y-\mu t)^2}/2t\sigma^2 dy$ Put ω = (y-$\mu t)/\sigma y^{\frac{1}{2}}$ we have

$$ce^{\alpha t} = x_0 e^{\mu t} \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{\sigma\omega\sqrt{t}} e^{-\frac{\omega^2}{2}} d\omega - k \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{-\frac{\omega^2}{2}} d\omega \qquad \text{......................... (3.4)}$$

Where a = $\frac{\log(k/x_0) - \mu t}{\sigma\sqrt{t}}$

Now $\frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{\sigma\omega\sqrt{t}} e^{-\frac{\omega^2}{2}} d\omega = e^{t\sigma^2/2} \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{-(\omega-\sigma\sqrt{t})^2/2}$ d ω = $e^{t\sigma^2/2}$ P[N($\sigma\sqrt{t}$,1) $\geq a] = e^{t\sigma^2/2}$ P[N(0 ,1) $\geq a - \sigma\sqrt{t}] = e^{t\sigma^2/2}$ P[N(0 ,1) $\leq -(a - \sigma\sqrt{t})] = e^{t\sigma^2/2}$ $\phi(\sigma\sqrt{t} - a)$ where N(m, v) is a normal random variable with mean m and variance v *and* ϕ *is the standard normal distributiion function* .

Thus from (2.4) we get $ce^{\alpha t} = x_0 e^{\mu t + t\sigma^2/2} \phi(\sigma\sqrt{t} - a) - k\phi(-a)$

Using $\mu + t\sigma^2 = \alpha$ *and letting* $b = -a$ c=$x_0\phi(\sigma\sqrt{t} + b)$-k$e^{-\alpha t}$ $\phi(b)$. .... (3.5)

where b = $\frac{\alpha t - t\sigma^2/2 - \log(k/x_0)}{\sigma\sqrt{t}}$

The option price formula given by (2.5) depends on the initial price of the stock x_0 , the option exercise time t , the option exercise price k , the discount (or interest rate) factor α , and the value σ^2. Note that for any value of σ^2, if the options are priced according to the formula of equation (2.5) Then no arbitrage is possible. Therefore price of a stock actually follows a Geometric Brownian Motion - That is X (t) = $x_0 e^{\gamma(t)}$ where $\gamma(t)$ is Brownian motion with parameter μ and σ^2. The formula (2.5) is known as Black Scholes Option Cost Valuation.

6. Refrences:

B.B. Mandelbrot and H.W. Van Ness, "Fractional Brownian motions, fractional noises and applications,"*SIAM Rev.,* Vol. 10, pp. 422-436, *oa.* 1968.

N. Tamas. Arbitrage Theorem and its Applications. “Club of Economics in Miskolc” TMP Vol. 1., pp. 27–32. 2002.

J. Audounet, G. Montseny, and B. Mbodje, Optimal models of fractional integrators and application to systems with fading memory, 1993, IEEE SMC's conference, Le Touquet(France).

J. Beran and N. Terrin, Testing for a change of the long-memory parameter, Biometrika 83 (1996), no. 3, 627{638).

S.M. Ross.Introduction to Probability Models.10th edition.Academic Press,San Diego,California.USA.

V.S. Borkar, Probability theory: an advanced course, Universitext, Springer, 1995.

Yu.A. Rozanov, Stationary random processes, Holden{Day, San Francisco, 1966.