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Topics in Stochastic Analysis and Applications to Finance

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Abstract

This research is concerned with the existence a relationship between Stochastic Differential Equations (SDEs), Backward Stochastic Differential Equations (BSDEs) and Partial Differential Equations (PDEs).

This research includes a major application of stochastic analysis, we study about applications to mathematical finance, and particularly, options as an example of financial derivatives and Black-Scholes method.

1 Introduction

Stochastic calculus is a branch of mathematics that operates on stochastic processes. It allows a consistent theory of integration to be defined for integrals of stochastic processes with respect to semi-martingales. It is used to model systems that behave randomly. These include stochastic differential equation (SDEs). The best-known stochastic process to which stochastic calculus is applied is the Brownian motion / Wiener process (named in honor of Norbert Wiener), which is used for modeling Brownian motion as described by Louis Bachelier in 1900 and by Albert Einstein in 1905 and other physical diffusion processes in space of particles subject to random forces. Since the 1970s, Brownian motion has been widely applied in financial mathematics and economics to model the evolution in time of stock prices, options, and bond interest rates. Stochastic analysis, involving analytical nature, which is an infinite dimensional analysis has become nowadays one of the most important and attractive fields due to its applications in different fields such as PDEs, differential geometry, finance, Malliavin calculus and potential theory.

We refer the reader to [1], [2], [4], and [5], for more information and applications.

2 Stochastic Calculus

2.1 Preliminaries in Measure Theory

We start this section by some information from measure theory and integration. More details and proofs can be found for example in [2], [1] and [4].

Definition 2.1 *Let* Ω *be a nonempty set and* $\mathcal{F}$ *be a collection of subsets of* Ω. *We say that* $\mathcal{F}$ *is a* $\sigma-$*field (*$\sigma-$*algebra) if:*

(i) $\Omega \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$,

(iii) $A_1, A_2, \ldots, A_n, \ldots \in \mathcal{F} \Rightarrow \bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$. In this case, the pair $(\Omega, \mathcal{F})$ is called a *measurable space*.

Definition 2.2 *The* $\sigma-$*algebra generated by all open sets in* $\mathbb{R}$ *is called the Borel* $\sigma-$*algebra and is denoted by* $\mathcal{B}(\mathbb{R})$. *Its elements are called Borel sets or Borel measurable sets.*

Definition 2.3 *Let* $(\Omega, \mathcal{F})$ *be a measurable space. A measure* $\mathbb{P}$ *on* $\mathcal{F}$ *is a function* $\mathbb{P} : \mathcal{F} \to [0, \infty)$ *with the properties:*

(i) $\mathbb{P}(\phi) = 0$,

(ii) if $A_i \in \mathcal{F} \ \ \forall j \geq 1$, and $A_i \cap A_j = \phi \ \ \forall i \neq j$, then

$$\mathbb{P}\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mathbb{P}(A_j).$$

The triplet $(\Omega, \mathcal{F}, \mathbb{P})$ is space called a *measure space* and the members of $\mathcal{F}$ are called *measurable sets*. If

$\mathsf{P}(\Omega)=1$, then P is called a *probability measure*, and $(\Omega,\mathsf{F},\mathsf{P})$ is a *probability space*. A measurable set A is called a *null set* if $\mathsf{P}(A)=0$.

Definition 2.4 *(i) Let* (Ω,F) *be a measurable space. A function* $f:\Omega\to\mathsf{R}$ *is called measurable if* $f^{-1}(B)\in\mathsf{F}\quad\forall B\in\mathsf{B}(\mathsf{R})$, *where* $\mathsf{B}(\mathsf{R})$ *is Borel* $\sigma-$*algebra, i.e.*

$$\{\omega\in\Omega\mid f(\omega)\in B\}=\{f\in B\}\in\mathsf{F}\quad\forall B\in\mathsf{B}(\mathsf{R}).$$

(ii) Let (Ω,F) be a measurable space. A function $f:\Omega\to\mathsf{R}$ is called a *random variable* if f is measurable.

2.2 Stochastic Processes

Definition 2.5 *A stochastic process is a collection* $X:[0,T]\times\Omega\to\mathsf{R}$ $\{X(t,\omega):(t,\omega)\in[0,T]\times\Omega\}$ *such that*

(i) for each $t, X(t,.)$ is R-valued random variable on Ω,

(ii) for each ω, $X(\cdot,\omega)$ is measurable $(t\to X(t,\omega))$ is called a *sample path*.

Thus a stochastic process $X(t,\omega)$ can be expressed as $X(t)(\omega)$ or simply as $X(t)$ or X_t.

Definition 2.6 *Let X be a random variable, i.e.* $X:\Omega\to\mathsf{R}$ *is measurable. Define the expectation of* X *by*

$$\mathsf{E}[X]=\int_\Omega X(\omega)d\mathsf{P}=\int_\Omega Xd\mathsf{P}.$$

The *variance* is $var(X)=\mathsf{E}[X-\mathsf{E}[X]]^2=\mathsf{E}[X^2]-\mathsf{E}[X]^2$.

Definition 2.7 *A stochastic process* $B(t,\omega)$ *is called a Brownian motion if it satisfies the following conditions:*

(i) $\mathsf{P}\{\omega\mid B(0,\omega)=0\}=1$,

(ii) For any $0\le s<t$, the random variable $B(t)-B(s)$ is normally distributed with mean 0 and variance $t-s$, *i.e.*, for any $a<b$

$$\mathsf{P}\{a\le B(t)-B(s)\le b\}=\frac{1}{\sqrt{2\pi(t-s)}}\int_a^b e^{\frac{-x^2}{2(t-s)}}dx.$$

(iii) $B(t,\omega)$ has independent increments, *i.e.*, for any $0\le t_0<t_1<t_2<\ldots<t_n$, the random variables $B(t_0), B(t_1)-B(t_0), B(t_2)-B(t_1),\ldots, B(t_n)-B(t_{n-1})$, are independent,

(iv) Almost all sample paths of $B(t,\omega)$ are continuous functions, i.e.

$$\mathsf{P}\{\omega\mid B(.,\omega)\text{is continuous}\}=1.$$

2.3 Martingales

First we start with the notion of conditional expectation.

Definition 2.8 *Let* $(\Omega,\mathsf{F},\mathsf{P})$ *be a probability space and* $X:\Omega\to\mathsf{R}$ *be an integrable random variable, i.e.* $\mathsf{E}[|X|]=\int_\Omega |X|d\mathsf{P}<\infty$. *Assume* $\mathsf{G}\subset\mathsf{F}$ *a sub* $\sigma-$*algebra. We say that* Y *is the conditional expectation of* X *with respect to* G, *and denote it by* $\mathsf{E}[X\mid\mathsf{G}]$, *if*

(i) Y is G- measurable,

(ii) $\int_E Xd\mathsf{P}=\int_E Yd\mathsf{P}\quad\forall E\in\mathsf{G}$.

Existence and uniqueness of conditional expectation is achieved by Radon-Nikodym theorem, a wellknown theorem in functional analysis.

Note that the conditional expectation $\mathsf{E}[X \mid \mathsf{G}]$ is random, while the expectation $\mathsf{E}[X]$ is deterministic.

We recall that a *filtration* on a probability space $(\Omega, \mathsf{F}, \mathsf{P})$ is a sequence $\{\mathsf{F}_t\}_{0 \le t \le T}$ of sub-sigma algebras of F such that for all $t > s$, $\mathsf{F}_s \subseteq \mathsf{F}_t$.

Definition 2.9 *A stochastic process* $\{X_t, t \ge 0\}$ *is called a martingale with respect to* $\{\mathsf{F}_t, t \ge 0\}$ *if:*

(i) X is adapted to $\{\mathsf{F}_t, t \ge 0\}$, i.e. X_t is $\mathsf{F}_t -$measurable for all $t \ge 0$.

(ii) $E[|X_t|] < \infty, \forall t \ge 0$.

(iii) $E[X_t \mid \mathsf{F}_s] = X_s, \forall t \ge s.$

2.4 Stochastic Integration

Let us first assume that $f : [0, \infty) \times \Omega \to \mathsf{R}$ has the form

$$f = \phi(t, \omega) = \sum_{j0} e_j(\omega) \chi_{[j \cdot 2^{-n}, (j+1)2^{-n})}(t),$$

where χ denotes the characteristic (indicator) function and n is a natural number. For such functions it is reasonable to define

$$\int_S^T \phi(t) dB(t) = \sum_{j0} e_j(\omega)[B_{t_{j+1}} - B_{t_j}](\omega), \tag{2.1}$$

where

$$t_k = t_k^{(n)} = \begin{Bmatrix} k \cdot 2^{-n} \text{ if } S \le k \cdot 2^{-n} \le T \\ S \text{if } k \cdot 2^{-n} < S \\ T \text{if } k \cdot 2^{-n} > T \end{Bmatrix}, \text{and} \quad B_j = B_{t_j}.$$

Without any further assumptions on the functions $e_j(\omega)$ this leads to difficulties. We slant then with the details in this issue.

Definition 2.10 *Let* $B_t(\omega)$ *be n-dimensional Brownian motion. Then we define* $\mathsf{F}_t = \mathsf{F}_t^{(n)}$ *to be the* $\sigma -$ *algebra generated by the random variables* $B_s(\cdot)$*;* $s \le t$*. In other words,* F_t *is the smallest* $\sigma -$ *algebra containing all sets of the form*

$$\{\omega : B_{t_1}(\omega) \in F_1, \Lambda\ , B_{t_k}(\omega) \in F_k\},$$

where $t_j \le t$ and $F_j \subset \mathsf{R}^n$ are Borel sets, $j \le k = 1, 2, \Lambda$. (We assume that all sets of measure zero are included in F_t).

Definition 2.11 *Let* $\mathsf{V} = \mathsf{V}(S, T)$ *be the class of functions*

$$f(t, \omega) : [0, \infty) \times \Omega \to \mathsf{R}$$

such that

(i) $(t, \omega) \to f(t, \omega)$ is $\mathsf{B} \times \mathsf{F}$ -measurable, where B denotes the Borel σ -algebra on $[0, \infty)$,

(ii) $f(t, \omega)$ is F_t -adapted,

(iii) $\mathsf{E}[\int_S^T f(t, \omega)^2 dt] < \infty.$

2.4.1 The Itô Integral

For functions $f \in \mathsf{V}$ we will now show how to define the Itô integral

$$\mathsf{I}[F](\omega) = \int_S^T f(t,\omega)dB_t(\omega),$$

where B_t is 1-dimensional Brownian motion.

The idea is natural: First we define $\mathsf{I}[\phi]$ for a simple class of functions ϕ. Then we show that each $f \in \mathsf{V}$ can be approximated (in an appropriate sense) by such $\phi's$ and we use this to define $\int fdB$ as the limit of $\int \phi dB$ as $\phi \to f$.

We now give the details of this construction: A function $\phi \in \mathsf{V}$ is called *elementary* if it has the form

$$\phi(t,\omega) = \sum_j e_j(\omega) \cdot \chi_{[t_j, t_{j+1})}(t).$$

Note that since $\phi \in \mathsf{V}$ each function e_j must be F_{t_j} − measurable.

For elementary functions $\phi(t,\omega)$ we define the integral according to (2.1), i.e.

$$\int_S^T \phi(t)dB_t(\omega) = \sum_{j0} e_j(\omega)[B(t_{j+1}) - B(t_j)](\omega). \tag{2.2}$$

Now we make the following important observation:

Lemma 2.1 *(Itô isometry)*

If $\phi(t,\omega)$ is bounded and elementary then

$$\mathsf{E}\left[(\int_S^T \phi(t,\omega)dB_t(\omega))^2 \right] = \mathsf{E}\left[\int_S^T \phi(t,\omega)^2 dt \right]. \tag{2.3}$$

The idea is now to use the isometry (2.3) to extend the definition from elementary functions to functions in V. We do this in several steps:

Step 1. Let $g \in \mathsf{V}$ be bounded and $g(\cdot,\omega)$ continuous for each ω. Then there exist elementary functions $\phi_n \in \mathsf{V}$ such that

$$\mathsf{E}\left[\int_S^T (g - \phi_n)^2 dt \right] \to 0 \quad as \quad n \to \infty.$$

Step 2. Let $h \in \mathsf{V}$ be bounded. Then there exist bounded functions $g_n \in \mathsf{V}$ such that $g_n(\cdot,\omega)$ is continuous for each ω and n, and

$$\mathsf{E}\left[\int_S^T (h - g_n)^2 dt \right] \to 0.$$

Step 3. Let $f \in \mathsf{V}$. Then there exists a sequence $\{h_n\} \subset \mathsf{V}$ such that h_n is bounded for each n, and

$$\mathsf{E}\left[\int_S^T (f - h_n)^2 dt \right] \to 0 \quad as \quad n \to \infty.$$

We are now ready to complete the definition of Itô integral

$$\int_S^T f(t,\omega)dB_t(\omega)$$

for $f \in \mathsf{V}$. If $f \in \mathsf{V}$ we choose, by Steps 1-3, elementary functions $\phi_n \in \mathsf{V}$ such that

$$\mathsf{E}\left[\int_S^T (f - \phi_n)^2 dt \right] \to 0.$$

Then we define

$$\mathsf{I}[f](\omega) := \int_S^T f(t,\omega)dB_t(\omega) := \lim_{n\to\infty} \int_S^T \phi_n(t,\omega)dB_t(\omega).$$

The limit exists as an element of $L^2(\Omega,\mathsf{F},\mathsf{P})$, since $\{\int_S^T \phi_n(t,\omega)dB_t(\omega)\}$ forms a Cauchy sequence in $L^2(\Omega,\mathsf{F},\mathsf{P})$.

Now using this definition by approximation via elementary processes we deduce with the help of Lemma 2.1 the following Itô isometry for elements of $\mathcal{V}(S,T)$.

Corollary 2.1 *(Itô isometry)*

$$\mathrm{E}\left[(\int_S^T f(t,\omega)dB_t)^2\right] = \mathrm{E}\left[\int_S^T f^2(t,\omega)dt\right] \quad \text{for all} \quad f \in \mathcal{V}(S,T). \tag{2.4}$$

Theorem 2.1 *Let* $f, g \in \mathcal{V}(S,T)$, $0 \quad S < U < T$, *and* c *be a constant. Then*

(i) $\int_S^T f dB_t = \int_S^U f dB_t + \int_U^T f dB_t$ for a.a. ω,

(ii) $\int_S^T (cf + g) dB_t = c\int_S^T f dB_t + \int_S^T g dB_t$ for a.a. ω,

(iii) $\mathrm{E}[\int_S^T f dB_t] = 0$,

(iv) $\int_S^T f dB_t$ is $\mathcal{F}_T -$ measurable.

2.4.2 Extensions of the Itô integral

Itô integral $\int f dB$ can be defined for a larger class of integrands f than $\mathcal{V}$. Let us expand this issue briefly. Assume that there exists an increasing family of $\sigma -$ algebras $\{\mathcal{H}_t; t \geq 0\}$ such that B is a martingale with respect to $\{\mathcal{H}_t, t \geq 0\}$.

Let us denote by $\mathcal{V}_{\mathcal{H}}^k(S,T)$ to the class of processes $f(t,\omega) \in \mathbb{R}^k$ satisfying:

(i) $(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel $\sigma -$ algebra on $[0,\infty)$,

(ii) f_t is $\mathcal{H}_t$-measurable $\forall t \geq 0$,

(iii) $\mathrm{E}[\int_S^T |f(s,\omega)|^2 ds < \infty] = 1$.

Since B is a martingale then $\mathcal{F}_t \subset \mathcal{H}_t, \quad \forall t \geq 0$.

Here k can be 1, in which case we simply write $\mathcal{V}_{\mathcal{H}}(S,T)$ for $\mathcal{V}_{\mathcal{H}}^1(S,T)$ and can be $m \times n$, which is the case when f is a matrix. If $m = 1$, we write $\mathcal{V}_{\mathcal{H}}^n(S,T)$ instead of $\mathcal{V}_{\mathcal{H}}^{1\times n}(S,T)$. If $\mathcal{H}_t = \mathcal{F}_t^{(n)} \quad \forall t \geq 0$, we write $\mathcal{V}^k(S,T) = \mathcal{V}_{\mathcal{F}^{(n)}}^k(S,T)$. Thus $\mathcal{V}^1(S,T) = \mathcal{V}(S,T)$, $\mathcal{V}^{m\times n} = \mathcal{V}^{m\times n}(0,\infty) = \bigcap_{T>0} \mathcal{V}^{m\times n}(0,T)$.

Definition 2.12 $\mathcal{W}_{\mathcal{H}}(S,T)$ *denotes the class of processes* $f(t,\omega) \in \mathbb{R}$ *satisfying*

(i) $(t,\omega) \to f(t,\omega)$ is $\mathcal{B} \times \mathcal{F}$-measurable, where $\mathcal{B}$ denotes the Borel $\sigma -$ algebra on $[0,\infty)$,

(ii) There exists an increasing family of $\sigma -$ algebras $\mathcal{H}_t; t \geq 0$ such that

a) B_t is a martingale with respect to $\mathcal{H}_t$ and

b) f_t is $\mathcal{H} -$ adapted,

(iii) $\mathrm{P}[\int_S^T f(s,\omega)^2 ds < \infty] = 1$.

Similarly to the notation for $\mathcal{V}$ we put $\mathcal{W}_{\mathcal{H}} = \bigcap_{T>0} \mathcal{W}(0,T)$, and in the matrix case we write $\mathcal{W}^{m\times n}(S,T)$ etc. If $\mathcal{H} = \mathcal{F}^{(n)}$ we write $\mathcal{W}(S,T)$ instead of $\mathcal{W}_{\mathcal{F}^{(n)}}(S,T)$ etc. If the dimension is clear from the context we sometimes drop the superscript and write $\mathcal{F}$ for $\mathcal{F}^{(n)}$ and so on.

Theorem 2.2 *(The Itô representation theorem)*

Let $F \in L^2(\Omega, \mathsf{F}_T^{(n)}, \mathsf{P})$. Then there exists a unique stochastic process $f(t,\omega) \in \mathsf{V}^n(0,T)$ such that

$$F(\omega) = \mathsf{E}[F] + \int_0^T f(t,\omega)dB(t). \tag{2.5}$$

More details on this fact as well as its proof can be found in [5, P. 52].

Theorem 2.3 *(Martingale representation theorem)*

Let $B(t) = (B_1(t),...,B_n(t))$ be an n-dimensional Brownian motion. Suppose M_t is an $\mathsf{F}_t^{(n)}-$ martingale (with respect to P) and that $M_t \in L^2(\Omega, \mathsf{F}, \mathsf{P})$ for all $t \geq 0$. Then there is a unique stochastic process $g(s,\omega)$ such that $g \in \mathsf{V}^{(n)}(0,t)$ for all $t \geq 0$ and

$$M_t(\omega) = \mathsf{E}[M_0] + \int_0^t g(s,\omega)dB(s) \quad a.s., \quad \forall t \geq 0. \tag{2.6}$$

3 Itô's Formula

Definition 3.1

Let B be a 1-dimensional Brownian motion on $(\Omega, \mathsf{F}, \mathsf{P})$. A 1-dimensional Itô process is a stochastic process X on $(\Omega, \mathsf{F}, \mathsf{P})$ of the form

$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB(s), \quad t \geq 0, \tag{3.1}$$

where $v \in \mathsf{W}_\mathsf{H}$, so that

$$\mathsf{P}\left[\int_0^t v(s,\omega)^2 ds < \infty, \text{for all } t \geq 0\right] = 1. \tag{3.2}$$

We also assume that u is H_t-adapted and

$$\mathsf{P}\left[\int_0^t |u(s,\omega)| ds < \infty, \text{for all } t \geq 0\right] = 1. \tag{3.3}$$

If X is an Itô process, equation (3.1) is sometimes written in the shorten differential form

$$dX_t = udt + vdB_t. \tag{3.4}$$

Theorem 3.1 *(Itô's formula)*

Let X be an Itô process given by

$$dX_t = udt + vdB_t.$$

Let $g(t,x) \in C^2([0,\infty) \times \mathsf{R})$. Then $Y_t = g(t,X_t)$ is again on Itô process, and

$$dY_t = \frac{\partial g}{\partial t}(t,X_t)dt + \frac{\partial g}{\partial x}(t,X_t)dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,X_t)dt.$$

4 Stochastic Differential Equations

A stochastic differential equation (SDE) is an equation of the form

$$X_t = X_0 + \int_0^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dB_s,$$

or in differential form

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t, \tag{4.1}$$

where $X_0 : \Omega \to \mathsf{R}^n$ is a random variable, $b:[0,T] \times \mathsf{R}^n \to \mathsf{R}^n$ and $\sigma:[0,T] \times \mathsf{R}^n \to \mathsf{R}^{n \times m}$ are measurable functions.

5 Backward Stochastic Differential Equations

5.1 Notation

Let $(\Omega, \mathsf{F}, \mathsf{P})$ be a complete probability space and W be a $d-$dimensional Brownian motion on this space. We will denote by $(\mathsf{F}_t)_{t\geq 0}$ the natural filtration of W.

We will work with two spaces of processes.

- Denote by $\mathsf{S}^2(\mathsf{R}^k)$ to the vector space formed by the processes Y, which are progressively measurable, take values in R^k.

S^2_c is the subspace of S^2 formed by continuous process. Two indistinguishable process in such spaces will be equal and we will use the same notations for the quotient spaces.

- Denote $\mathsf{M}^2(\mathsf{R}^{k\times d})$ to the set of those processes Z, progressively measurable, with values in $\mathsf{R}^{k\times d}$,

where if $z \in \mathsf{R}^{k\times d}$. Denote $\mathsf{M}^2(\mathsf{R}^{k\times d})$ as the set of equivalent classes of $\mathsf{M}^2(\mathsf{R}^{k\times d})$.

Will be often omit R^k and $\mathsf{R}^{k\times d}$; so in particular the space S^2, S^2_c and M^2 are Banach spaces for the preceding defined norms. We denote by B^2 to the Banach space $\mathsf{S}^2(\mathsf{R}^k)\times\mathsf{M}^2(\mathsf{R}^{k\times d})$.

In this section we are given a random mapping f defined on $[0,T]\times\Omega\times\mathsf{R}^k\times\mathsf{R}^{k\times d}$ with values in R^k such that, for any $(y,z)\in\mathsf{R}^k\times\mathsf{R}^{k\times d}$, the process $\{f(t,y,z)\}_{0\leq t\leq T}$ is progressively measurable. We are given also an F_T-random variable ξ, taking its values in R^k. We want to solve the following backward stochastic differential equation (BSDE):

$$dY_t = -f(t,Y_t,Z_t)dt + Z_t dW_t, \quad 0\leq t\leq T, \qquad Y_T=\xi, \tag{5.1}$$

or, equivalently, in the integral form:

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dW_s, \quad 0\leq t\leq T. \tag{5.2}$$

The function f is called the *generator* of BSDE (5.1) and ξ is the *terminal value* or *terminal condition*. See Definition 5.1 below for precise definition of the solution of BSDE (5.1).

Before introducing the notion of solution of BSDEs, let us discuss briefly a simple case of (5.1).

Example 5.1 *Consider the following differential equation for* $t\in[0,T]$,

$$dY_t = 0, \qquad Y_T = \xi. \tag{5.3}$$

If we treat this equation as an ODE, i.e. $\xi\in\mathsf{R}$, then it possesses the unique solution $Y_t=\xi$. But if we treat this equation as an SDE, i.e. $\xi\in L^2(\Omega,\mathsf{F}_T,\mathsf{P})$, then, in general, no F_t-adapted solution exists because the only candidate solution $Y_t=\xi$ is not necessarily adapted to the underlying filtration

Let us reformulate the differential equation (5.3) so that it makes sense in a stochastic setting. Set $Y_t=\mathsf{E}[\xi\,|\,\mathsf{F}_t]$, where F_t is assumed to be the Brownian filtration. By the martingale representation theorem (Theorem 2.3), there exists a process Z in M^2, such that

$$Y_t = Y_0 + \int_0^t Z_s dW_s$$

or, in differential form, this becomes

$$dY_t = Z_t dW_t, \quad Y_T = \xi.$$

The solution of this backward SDE should then be given by a pair (Y_t, Z_t).

Definition 5.1 *A solution of BSDE (5.1) is a couple of processes* (Y, Z) *satisfying:*

1. Y and Z are progressively measurable with values respectively in $\mathbb{R}^k$ and $\mathbb{R}^{k\times d}$;
2. $\mathbb{E}\int_0^T (|Y_s|^2 + |Z_s|^2) ds < \infty$;
3. $\mathbb{P}$ -a.s., we have:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \le t \le T.$$

6 Applications

6.1 Application to PDEs

Consider the PDE in $\mathbb{R}$

$$\begin{aligned} -v_t(t,x) - \mathbb{L}v(t,x) - f(t,x) = 0, \\ v(T,x) = g(x), \end{aligned} \quad (6.1)$$

where $\mathbb{L}$ is the generator of the forward SDE

$$\begin{aligned} dX_s &= b(X_s)ds + \sigma(X_s)dW_s, \\ X_t &= x \in \mathbb{R}, \end{aligned} \quad (6.2)$$

given by

$$\mathbb{L}h(x) = b(x)h'(x) + \frac{1}{2}\sigma^2(x)h''(x), \quad h \in C^2(\mathbb{R}). \quad (6.3)$$

Here W is a Brownian motion in $\mathbb{R}$, and b, $\sigma : \mathbb{R} \to \mathbb{R}$ are Lipschitz mappings.

Theorem 6.1 *Assume* $v \in C^{1,2}([0,T] \times \mathbb{R}, \mathbb{R})$ *is a solution of (6.1) such that for some constant* $C > 0$

$$|v(t,x)| + \left|\frac{\partial v}{\partial x}(t,x)\sigma(x)\right| \le C(1 + |x|^p) \quad (6.4)$$

for all $(t,x) \in [0,T] \times \mathbb{R}$, for some $p \ge 1$. Then, $v(t,x)$ can be represented as

$$v(t,x) = \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}) ds + g(X_t^{t,x})\right],$$

where $X_s^{t,x}, s \ge t$ is the solution of (6.2).

Proof. Let $Y_s = v(s, X_s^{t,x})$. From Itô's formula and (6.1), we deduce

$$dY_s = \frac{\partial v}{\partial s}(s, X_s^{t,x})ds + \frac{\partial v}{\partial y}(s, X_s^{t,x})b(X_s^{t,x})ds + \frac{\partial v}{\partial y}(s, X_s^{t,x})\sigma(X_s^{t,x})dW_s$$

$$+\frac{1}{2}\frac{\partial^2 v}{\partial y^2}(s,X_s^{t,x})\sigma^2(X_s^{t,x})ds$$

$$=-f(s,X_s^{t,x})ds+\frac{\partial v}{\partial y}(s,X_s^{t,x})\sigma(X_s^{t,x})dW_s,$$

by using (6.1). Therefore

$$Y_t=Y_T+\int_t^T f(s,X_s^{t,x})ds-\int_t^T\frac{\partial v}{\partial y}(s,X_s^{t,x})\sigma(X_s^{t,x})dW_s,$$

and since

$$Y_T=v(T,X_T^{t,x})=g(X_T^{t,x}),$$

we have

$$v(t,x)=\mathsf{E}[Y_t]=\mathsf{E}\left[g(X_T^{t,x})+\int_t^T f(s,X_s^{t,x})ds\right].$$

The expectation of the stochastic integral vanishes with the help of (6.4). This completes the proof.

We now study a most general PDE than (6.1) in which we deal with a nonlinear are when f in values v and its derivative.

Theorem 6.2 *Let* $v\in C^{1,2}$ *be a classical solution of*

$$\begin{aligned}&-v_t(t,x)-\mathsf{L}v(t,x)-f(t,x,v(t,x),\sigma(x)v_x(t,x))=0\\&v(T,x)=g(x).\end{aligned}\tag{6.5}$$

Assume that v satisfies v_x satisfy (6.4) in Theorem 6.1 (with $p=1$). Let $X_s=X_s^{t,x}, s\geq t,$ denote the solution of (6.2) assuming that b and σ are Lipschitz. Define

$$Y_s=v(s,X_s),\qquad Z_s=\sigma(X_s)v_x(s,X_s).$$

Then (Y,Z) solves the BSDE

$$\begin{aligned}&-dY_s=f(s,X_s,Y_s,Z_s)ds-Z_s dW_s,\\&Y_T=g(X_T),\end{aligned}\tag{6.6}$$

furnished over the completed filtration $\sigma\{W_r-W_t \mid t\leq r\leq s\}\vee\mathsf{N}$, $s\geq t$, where N is the collection of P-null sets.

Proof. Apply Itô's formula to $Y_s=v(s,X_s)$ for $t\leq s\leq T$, to get by using also (6.2),

$$dY_s=\frac{\partial v}{\partial s}(s,X_s)ds+\frac{\partial v}{\partial x}(s,X_s)dX_s+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}(s,X_s)dX_s dX_s$$

$$=\frac{\partial v}{\partial s}(s,X_s)ds+\frac{\partial v}{\partial x}(s,X_s)b(X_s)ds+\frac{\partial v}{\partial x}(s,X_s)\sigma(X_s)dW_s$$

$$+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}(s,X_s)\sigma^2(X_s)ds.\tag{6.7}$$

But from (6.5),

$$-v_s(s,X_s)-\mathsf{L}v(s,X_s)-f(s,x,v(s,X_s),\sigma(X_s)v_x(s,X_s))=0$$

or in particular

$$-\frac{\partial v}{\partial s}(s,X_s)-b(X_s)\frac{\partial v}{\partial x}(s,X_s)-\frac{1}{2}(\sigma^2)(X_s)\frac{\partial^2 v}{\partial x^2}(s,X_s)$$

$$-f(s,X_s,v(s,X_s),\sigma(X_s)\frac{\partial v}{\partial x}(s,X_s))=0.$$

Thus (6.7) becomes

$$dY_s=-f(s,X_s,v(s,X_s),\sigma(X_s)\frac{\partial v}{\partial x}(s,X_s))ds+\frac{\partial v}{\partial x}(s,X_s)\sigma(X_s)dW_s$$

$$=-f(s,X_s,Y_s,Z_s)ds+Z_s dW_s,\ \text{ with } Z_s=\frac{\partial v}{\partial x}(s,X_s)\sigma(X_s).$$

Remark 6.1 *Note that* $Y_t=v(t,X_t^{t,x})=v(t,x)$ *is deterministic, so is* $Z_s=\frac{\partial v}{\partial x}(t,X_t)\sigma(X_t)=\frac{\partial v}{\partial x}(t,x)\sigma(x)$, *since they are* $\sigma\{0\}=\{\Omega,\phi\}$ *measurable*.

6.2 Application to Mathematical Finance

6.2.1 The Market, Portfolio and Arbitrage

Definition 6.1 *a) Let* $F_t^{(m)}$ *be the natural filtration of an m-dimensional Brownian motion* $(B_1,\Lambda\ ,B_m)$. *A market is an* $F_t^{(m)}-$ *adapted (n+1)-dimensional Itô process* $X(t)=(X_0(t),X_1(t),\Lambda\ ,X_n(t)),\ 0\le t\le T$, *which we will assume has the form*

$$dX_0(t)=\rho(t,\omega)X_0(t)dt;\quad X_0(0)=1, \tag{6.8}$$

and

$$dX_i(t)=\mu_i(t,\omega)dt+\sum_{j=1}^{m}\sigma_{ij}(t,\omega)dB_j(t)$$

$$=\mu_i(t,\omega)dt+\sigma_i(t,\omega)dB(t),\quad X_i(0)=x_i\in R,$$

where σ_i is row number i of the $n\times m$ matrix $[\sigma_{ij}],\ 1\le i\le n\in N$.

b) The market $\{X(t)\}_{t\in[0,T]}$ is called *normalized* if $X_0(t)\equiv 1$.

c) A *portfolio* in the market $\{X(t)\}_{t\in[0,T]}$ is an (n+1)-dimensional $(t,\omega)-$measurable and $F_t^{(m)}-$ adapted stochastic process

$$\theta(t,\omega)=(\theta_0(t,\omega),\theta_1(t,\omega),...,\theta_n(t,\omega));\quad 0\le t\le T. \tag{6.9}$$

d) The *value* at time t of a portfolio $\theta(t)$ is defined by

$$V(t,\omega)=V^{\theta}(t,\omega)=\theta(t)\cdot X(t)=\sum_{i=0}^{n}\theta_i(t)X_i(t) \tag{6.10}$$

where $\cdot$ denotes inner product in R^{n+1}.

e) The portfolio $\theta(t)$ is called *self-financing* if

$$\int_0^T\left(|\theta_0(s)\rho(s)X_0(s)+\sum_{i=1}^{n}\theta_i(s)\mu_i(s)|+\sum_{j=1}^{m}\left[\sum_{i=1}^{n}\theta_i(s)\sigma_{ij}(s)\right]^2\right)ds<\infty\quad a.s. \tag{6.11}$$

and

$$dV(t)=\theta(t)\cdot dX(t) \tag{6.12}$$

i.e.

$$V(t) = V(0) + \int_0^t \theta(t) \cdot dX(t), \text{for} \quad t \in [0,T]. \tag{6.13}$$

Remark 6.2 *Note that we can always make the market normalized by defining*

$$\overline{X}_i(t) = X_0(t)^{-1} X_i(t), \quad 1 \le i \le n. \tag{6.14}$$

The market

$$\overline{X}(t) = (1, \overline{X}_1(t), \ldots, \overline{X}_n(t))$$

is called the normalization of $X(t)$.

Definition 6.2 *A portfolio $\theta(t)$ which satisfies (6.11) and which is self-financing is called admissible if the corresponding value process $V^\theta(t)$ is (t,ω) a.s. lower bounded, i. e. there exists $K = K(\theta) < \infty$ such that*

$$V^\theta(t,\omega) - K, \text{for a.a.}(t,\omega) \in [0,T] \times \Omega. \tag{6.15}$$

Definition 6.3 *An admissible portfolio $\theta(t)$ is called an arbitrage (in the market $\{X_t\}_{t\in[0,T]}$) if the corresponding value process $V^\theta(t)$ satisfies $V^\theta(0) = 0$ and*

$$V^\theta(T) 0 \quad \text{a.s. and} \quad \mathrm{P}[V^\theta(T) > 0] > 0.$$

In other words, $\theta(t)$ is an arbitrage if it gives an increase in the value from time $t = 0$ to time $t = T$ a.s., and a strictly positive increase with positive probability. So $\theta(t)$ generates a profit without any risk of losing money.

The following theorem is due to Dudley (1977), [3].

Theorem 6.3 *Let F be an $\mathcal{F}_T^{(m)}$-measurable random variable and let $B(t)$ be an m-dimensional Brownian motion. Then there exists $\phi \in \mathcal{W}^m = \mathcal{W}^{1\times m}_{\mathcal{F}^{(m)}_\cdot}(0,\infty) = \bigcap_{T>0} \mathcal{W}^{1\times m}_{\mathcal{F}^{(m)}_\cdot}(0,T)$, such that*

$$F(\omega) = \mathrm{E}[F] + \int_0^T \phi(T,\omega) dB(t). \tag{6.16}$$

Note that ϕ is not unique because F is not assumed to be square integrable.

This implies that for any constant z there exists $\phi \in \mathcal{W}^m$ such that

$$F(\omega) = z + \int_0^T \phi(T,\omega) dB(t).$$

Thus, if we let $m = n$ and interprete $B_1(t) = X_1(t), \ldots, B_n(t) = X_n(t)$ as prices, and put $X_0(t) \equiv 1$, this means that we can, with any initial fortune z, generate any $\mathcal{F}_T^{(m)}$ $-$measurable final value $F = V(T)$, as long as we are allowed to choose the portfolio ϕ freely from $\mathcal{W}^m$. This again underlines the need for some extra restriction on the family of portfolios allowed, like condition (6.15). This sentence is quoted from [5, P. 267].

Definition 6.4 *An equivalent measure Q to P $(P \sim Q)$ such that the normalized process $\{\overline{X}(t)\}_{t\in[0,T]}$ is a (local) martingale with respect to Q is called an equivalent (local) martingale measure.*

Proposition 6.1 *Suppose that a process $u(t,\omega) \in \mathcal{V}^m(0,T)$ satisfies the condition*

$$\mathrm{E}\left[\exp\left(\frac{1}{2}\int_0^T u^2(s,\omega) ds\right)\right] < \infty. \tag{6.17}$$

Define the measure $Q = Q_u$ on $\mathcal{F}_T^{(m)}$ by

$$dQ(\omega)=\exp\left(\int_0^T u(t,\omega)dB(t)-\frac{1}{2}\int_0^T u^2(t,\omega)dt\right)dP(\omega). \tag{6.18}$$

Then

$$\hat{B}(t):=\int_0^T u(s,\omega)ds+B(t) \tag{6.19}$$

is an $\mathcal{F}_t^{(m)}-$ martingale (and hence an $\mathcal{F}_t^{(m)}-$ Brownian motion) with respect to Q, and any $F\in L^2(\Omega,\mathcal{F}_T^{(m)},Q)$ has a unique representation

$$F(\omega)=E_Q[F]+\int_0^T \phi(t,\omega)d\hat{B}(t), \tag{6.20}$$

where $\phi(t,\omega)$ is an $\mathcal{F}_t^{(m)}-$adapted, $(t,\omega)-$measurable R^m-valued process, such that

$$E_Q\left[\int_0^T \phi^2(t,\omega)dt\right]<\infty. \tag{6.21}$$

For the proof of this proposition we refer the reader to [6, Proposition 17.1].

Remark 6.3 *From now on we assume that there exists a (not necessarily unique) process* $u(t,\omega)\in V^m(0,T)$ *satisfying*

$$\sigma(t,\omega)u(t,\omega)=\mu(t,\omega)-\rho(t,\omega)\hat{X}(t,\omega) \quad a.a.(t,\omega) \tag{6.22}$$

and

$$E\left[\exp\left(\frac{1}{2}\int_0^T u^2(s,\omega)ds\right)\right]<\infty, \tag{6.23}$$

where $\hat{X}(t,\omega)=(X_1(t,\omega),\ldots,X_n(t,\omega))$, and we let $Q=Q_u$ and $\hat{B}$ be as in (6.18), (6.19), as described in Proposition 6.1. This guarantees that the market $\{X(t)\}_{t\in[0,T]}$ has no arbitrage.

6.3 Option Pricing

Definition 6.5 *a) A (European) contingent $T-$claim (or just a $T-$claim or claim) is a lower bounded $\mathcal{F}_T^{(m)}-$measurable random variable $F(\omega)$.*

b) We say that the claim $F(\omega)$ is *attainable* (in the market $\{X(t)\}_{t\in[0,T]}$) if there exists an admissible portfolio $\theta(t)$ and a real number z such that

$$F(\omega)=V_z^\theta:=z+\int_0^T \theta(t)dX(t) \quad a.s.$$

and such that

$$\overline{V}^\theta(t)=z+\int_0^t \xi(s)\sum_{i=1}^n \theta_i(s)\sigma_i(s)d\hat{B}(s), \quad 0\le t\le T,$$

is a $Q-$martingale.

If such a $\theta(t)$ exists, we call it a *replicating* or *hedging portfolio* for F.

c) The market $\{X(t)\}_{t\in[0,T]}$ is called complete if every bounded T-claim is attainable.

Remark 6.4 *A market $\{X(t)\}$ is complete if and only if there is one and only one equivalent martingale measure for the normalized market $\{\overline{X}(t)\}$.*

6.3.1 European Options

Let $F(\omega)$ be a $T-$ claim. A European option on the claim F is a guarantee to be paid the amount $F(\omega)$ at time $t=T>0$. How much would you be willing to pay at time $t=0$ for such a guarantee? You could argue as follows:

If *I* - the buyer of the option -pay the price y for this guarantee, the *I* it must be possible to hedge to time T a value $V_{-y}^{\theta}(T,\omega)$ which, if the guaranteed payoff $F(\omega)$ is added, gives me a nonnegative result:

$$V_{-y}^{\theta}(T,\omega)+F(\omega)0 \quad a.s.$$

Thus the maximal price $p=p(F)$ the buyer is willing to pay is

$$p(F)=\sup\{y: \text{there exists an admissible portfolio } \theta \text{such that}$$
$$V_{-y}^{\theta}(T,\omega):=-y+\int_0^T\theta(s)dX(s)-F(\omega) \quad a.s.\}.$$

On the other hand, the seller of this guarantee could argue as follows:

If *I*- the seller -receive the price z for this guarantee, then *I* can use this as the initial value in an investment strategy. With this initial fortune it must be possible to hedge to time T a value $V_z^{\theta}(T,\omega)$ which is not less than the amount $F(\omega)$ that *I* have promised to pay to the buyer:

$$V_z^{\theta}(T,\omega) \quad F(\omega) \quad a.s.$$

Thus the minimal price $q=q(F)$ the seller is willing to accept is

$$q(F)=\inf\{z: \text{there exists an admissible portfolio } \theta \text{such that}$$
$$V_z^{\theta}(T,\omega):=z+\int_0^T\theta(s)dX(s) \quad F(\omega) \quad a.s.\}.$$

Definition 6.6 *If $p(F)=q(F)$ we call this common value the price (at $t=0$) of the (European) $T-$contingent claim $F(\omega)$.*

Two important examples of European contingent claims are

a) *the European call,* where

$$F(\omega)=(X_i(T,\omega)-K)^+$$

for some $i\in\{1,2,\Lambda\ ,n\}$ and some $K>0$. This option gives the owner the right to buy one unit of security number i at the specified price K at time T. So if $X_i(T,\omega)>K$ at time T, the owner of the option will exercise as he/she will obtain a payoff $X_i(T,\omega)-K$ at time $t=T$, while if $X_i(T,\omega)\le K$ then the owner will not exercise his option and the payoff is 0.

b) Similarly, *the European put* option gives the owner the right to sell one unit of security number i at a specified price K at time T. This option gives the owner the payoff

$$F(\omega)=(K-X_i(T,\omega))^+.$$

Example 6.1 *Suppose the market is given by*

$$X(t)=(1+B(t))\in \mathsf{R}^2, \quad t\in[0,T].$$

We may ask whether the claim

$$F(\omega)=B^2(T,\omega)$$

is attainable. We seek an admissible portfolio $\theta(t)=(\theta_0(t),\theta_1(t))$ and a real number z such that

$$F(\omega)=B^2(T,\omega)=z+\int_0^T\theta(t)\cdot dX(t)=z+\int_0^T\theta_1(t)dB(t). \tag{6.24}$$

By Itô's formula we see that

$$B^2(T,\omega) = T + \int_0^T 2B(t)dB(t).$$

We conclude that

$$z = T, \quad \theta_1(t) = 2B(t)$$

do the job (6.24). Then we choose $\theta_0(t)$ to make the portfolio $\theta(t)$ self-financing. For this we need that

$$V_z^\theta(t) = z + \int_0^t \theta(s)\cdot dX(s) = \theta(t)\cdot X(t)$$

i.e.

$$T + \int_0^t 2B(s)dB(s) = \theta_0(t) + 2B^2(t).$$

So we choose $\theta_0(t) = T + \int_0^t 2B(s)dB(s) - 2B^2(t)$. Then $\theta(t) = (\theta_0(t), \theta_1(t))$ is an admissible portfolio which replicates F and hence F is attainable.

Theorem 6.4 *(The generalized Black-Scholes formula)*
Suppose $X(t) = (X_0(t), X_1(t))$ is given by

$$dX_0(t) = \rho(t)X_0(t)dt, \quad X_0(0) = 1, \tag{6.25}$$

$$dX_1(t) = \alpha(t,\omega)X_1(t)dt + \beta(t)X_1(t)dW(t), \quad X_1(0) = x_1 > 0, \tag{6.26}$$

where W is a Brownian motion in R, $\rho(t), \beta(t)$ are deterministic, and

$$\mathsf{E}\left[\exp\left(\frac{1}{2}\int_0^T \frac{(\alpha(t,\omega)-\rho(t))^2}{\beta(t)^2}dt\right)\right] < \infty. \tag{6.27}$$

Then the market $\{X(t)\}$ is complete and the price at time $t = 0$ of the European $T-$ claim $F(\omega) = f(X_1(T,\omega))$ where $E_Q[f(X_1(T,\omega))] < \infty$ is

$$p = \frac{\xi(T)}{\delta\sqrt{2\pi}}\int_{\mathsf{R}} f\left(x_1 \exp\left[y + \int_0^T (\rho(s) - \frac{1}{2}\beta^2(s))ds\right]\right)\exp\left(-\frac{y^2}{2\delta^2}\right)dy \tag{6.28}$$

where

$$\xi(T) = \exp\left(-\int_0^T \rho(s)ds\right) \quad \text{and} \quad \delta^2 = \int_0^T \beta^2(s)ds. \tag{6.29}$$

Note that the solution of (6.26) is

$$X_1(t) = X_0(t)\exp\left(\int_0^t \beta(s,\omega)dB(s) + \int_0^t (\alpha(s,\omega) - \frac{1}{2}\beta^2(s,\omega))ds\right). \tag{6.30}$$

Corollary 6.1 *(The classical Black-Scholes formula)*
a) Suppose $X(t) = (X_0(t), X_1(t))$ is the classical Black-Scholes market

$$dX_0(t) = \rho X_0(t)dt, \quad X_0(0) = 1,$$

$$dX_1(t) = \alpha X_1(t)dt + \beta X_1(t)dW(t), \quad X_1(0) = x_1 > 0,$$

where $\rho, \alpha, \beta \neq 0$ are constants. Then the price p at time 0 of the European call option, with payoff

$$F(\omega) = (X_1(T,\omega) - K)^+ \tag{6.31}$$

where $K > 0$ is a constant (the exercise price) is

$$p = x_1\Phi(\eta + \frac{1}{2}\beta\sqrt{T}) - Ke^{-\rho T}\Phi(\eta - \frac{1}{2}\beta\sqrt{T}), \tag{6.32}$$

where

$$\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{1}{2}x^2}dx \quad y \in \mathsf{R}, \tag{6.33}$$

is the standard normal distribution function.

b) The replicating portfolio $\theta(t)=(\theta_0(t),\theta_1(t))$ for this claim F in (6.31) is given by

$$\theta_1(t,\omega)=\Phi\left(\beta^{-1}(T-t)^{-\frac{1}{2}}(\ln\frac{X_1(T)}{K}+\rho(T-t)+\frac{1}{2}\beta^2(T-t))\right) \tag{6.34}$$

with $\theta_0(t,\omega)$ determined and $V^{\theta}(0)=p.$

The above theorem (Theorem 6.4) and corollary (Corollary 6.1) are fundamental in mathematical finance. Their proof can be found in many text books such as [5, Chapter 6].

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