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The State-Space Model for the Double Heston Model written on two underlying assets

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Abstract

In this study, we look at the derivation of the state space model that is applied in non-linear filtering methods. The state-space model is for the double Heston Model written on two underlying assets. The non-linear filtering methods include unscented Kalman filter, extended Kalman filter and particle filter. This study extends work of Namundjebo (2016) who looked at the application of non-linear filtering methods to the Double Heston Model. We extend this work to the derivation of the state space model of the Double Heston Model with two underlying assets. The two underlying assets are assumed to be uncorrelated.

Keywords: non-linear filtering, unscented Kalman filter, extended Kalman filter, Kalman filter, particle filter, state-space model.

1. Introduction

The Heston Model which is named after Steven Heston, is an example of a stochastic volatility model. A stochastic volatility model refers to a model where the volatility of the asset returns are driven by a stochastic variance process. State space models have a wide range of applications including stochastic volatility models in finance.

Filtering is engineering terminology for extracting information about a signal from partial and noisy observations. Filtering can be used to estimate a dynamic system's internal states given that the system has a series of current and past noisy observations. The observation variables are observable unlike the system's states which are unobservable. The system's states conditional probability distribution can then be estimated using the filtering approach

We begin with the prediction step. Suppose that the vector x_k represents the current system's state at the current time k, and the prediction of x_k at time k is given by $\hat{x}_{k|k-1}$. An assumption is made that the previous estimates $\hat{x}_{k-1}$ are known, they are used to predict the state vector $\hat{x}_{k|k-1}$.

In the update step, the predicted states $\hat{x}_{k|k-1}$ are combined with the current observations y_k to estimate the current states $\hat{x}_{k|k}$. Given that the observations are noisy, we are interested in the best estimate $\hat{x}_{k|k}$ of x_k that minimizes the error, $x_k - \hat{x}_{k|k}$. This is done by recursion at each time step k.

A Kalman filter is an example of an optimal filtering method which is applicable in the field of science, engineering and finance. It can be used for the estimation of a model's parameters, when the model relies on non-observable data. It is considered easy to understand with little computational burdens. The Kalman filter is also ideal when a large volume of information must be taken into account, because it is very fast. In finance it can be used in hedging under partial observation, volatility estimation, optimal asset allocation, etc.

There are two basic building blocks of a Kalman Filter, the measurement equation and the transition equation. The measurement equation relates an unobserved variable to an observable variable. The transition equation is based on a model that allows the unobserved variable to change through time. The method requires first of all that the model is expressed on a state-space form. A state-space model is characterized by a measurement equation and a transition equation.

The Kalman filter is however only limited to linear models with Gaussian noises. Some non-linear filtering methods that are applicable to non-linear systems include the extended Kalman filter and the unscented Kalman filter. Particle filters can be applied to non-linear models with non-Gaussian noises.

A discrete dynamical system is considered with unobservable state vector x_k, for $k = 1, 2, \ldots$, where k denotes time

$$x_k = f_k(x_{k-1}, w_k) \quad (1.0.1)$$

and f_k is a possibly non-linear and time-dependent function that represents the evolution of the state process x_k. The state process is driven by noise denoted by w_k.

Suppose that an observable vector y_k at time k is also given such that:

$$y_k = h_k(x_k, v_k) \quad (1.0.2)$$

where h_k is a possibly non-linear and time-dependent function that defines the measurement y_k. The observations noise is denoted by v_k. The state process in Equation 1.0.1 is called the state transition equation and the observation process in 1.0.2 is called measurement equation. In order to obtain an estimate for the unobservable state x_k at a given time k given that we have all observations up to time k, The Bayes rule can be used to compute the following

$$p(x_k|y_{1:k}) = \frac{p(y_k|x_k)p(x_k|y_{1:k-1})}{p(y_k|y_{1:k-1})} \quad (1.0.3)$$

where $p(.)$ denotes probability density, $p(y_k|x_k)$ is the measurement probability or the likelihood function of the observation y_k given a state x_k.

In addition:

$$p(y_k|y_{1:k-1}) = \int p(y_k|x_k)p(x_k|y_{1:k-1})dx_k$$

and

$$p(x_k|y_{1:k-1}) = \int p(x_k|x_{k-1})p(x_{k-1}|y_{1:k-1})dx_{k-1} \quad (1.0.4)$$

$p(x_k|y_{1:k-1})$ is the probability density of the current state x_k conditioned on the measurements up to the time step $k-1$. This probability is obtained in the prediction step.

$p(x_k|y_{1:k})$ is the probability density of the current state x_k conditioned on all the current and past measurements. This probability is computed in the update step.

2. Filtering Techniques

2.1 Kalman Filter

Kalman filter is only optimal for linear systems.

Given that the state function f_k from Equation 1.0.1 and the measurement function h_k from the Equation 1.0.2 are linear and their corresponding noise w_k and v_k respectively, are normally distributed and additive. Equation 1.0.1 can be expressed as

$$x_k = M_k x_{k-1} + w_k \quad (2.1.1)$$

and Equation 1.0.2 becomes

$$y_k = H_k x_k + v_k \quad (2.1.2)$$

The matrix M_k is assumed to be known, it defines the state transition evolution, and the matrix H_k defines the measurement process which is also assumed to be known. An assumption is made that the state noise $w_k \sim N(0, Q_k)$ and the measurement noise $v_k \sim N(0, R_k)$ are uncorrelated Gaussian random variables. In addition, w_k, v_k are independent of x_k, y_k respectively.

By substituting x_k and y_k from Equations 2.1.1 and 2.1.2 in Equations 1.0.3 and 1.0.4, the computations result in the Kalman filtering algorithm where the distributions are given as

$$p(x_k|x_{k-1}) \sim N(M_k x_{k-1}, Q_k),$$
$$p(y_k|x_k) \sim N(H_k x_k, R_k).$$

The objective is to find an estimate of the state vector x_k given the observations y_k. An estimate of the state vector x_k is obtained from the past estimated states, $\hat{x}_{k-1}$ in the prediction step. We have that

$$\hat{x}_{k|\,k-1} = M_k \hat{x}_{k-1}$$

$k|\,k-1$ represents the estimated state of the state x_k using previous $(k-1)$ estimated states.

$k|\,k$ represents the estimates of x_k using estimated states at $k|\,k-1$.

The computation of the past states is conducted by making use of the expectation of x_k given in Equation 2.1.1.

The error in estimation error is obtained from

$$e_k^- = x_k - \hat{x}_{k|\,k-1}$$

and the estimate error covariance

$$P_k^- = [e_k^- e_k^{-T}]$$

The prediction of the observations is computed from

$$\hat{y}_k = H_k \hat{x}_{k|\,k-1}$$

The estimate of $\hat{x}_{k|\,k}$ is obtained from $\hat{x}_{k|\,k-1}$ and a measurement residual weighted by Kalman gain K_k in the update step as

$$\hat{x}_{k|\,k} = \hat{x}_{k|\,k-1} + K_k(y_k - \hat{y}_k)$$

The measurement residual is computed from $b_k = y_k - \hat{y}_k$. The estimate error is

$$e_k = x_k - \hat{x}_{k|\,k}$$

and the estimate error covariance

$$P_k = [e_k e_k^T]$$

The Kalman gain K_k is an averaging factor which is key in the Kalman filter. Since the predicted states $\hat{x}_{k|\,k-1}$ and $\hat{y}_k$ are known, the value of the Kalman gain K_k is set so that it minimizes the variance of e_k. We will always have $0 \leq K_k \leq 1$.

However, some systems can be more complex and non-linear, where the nonlinearity can be in the states process or in the measurements process or both.

Kalman filter is only optimal for linear systems. Therefore the need for nonlinear filters. Next we discuss an extension of the Kalman filter known as the extended Kalman filter which can handle non-linear Gaussian systems.

2.2 Extended Kalman Filter

This is an extension of the optimal Kalman filter and is used where the dynamical systems are non-linear.

Consider a case where the state transition function f_k and the observation function h_k given in Equations 1.0.1 and 1.0.2 respectively are both non-linear and their corresponding noises are uncorrelated Gaussian random variables, we have that $w_k \sim N(0, Q_k)$ and $v_k \sim N(0, R_k)$. Given that the densities in 1.0.3 and 1.0.4 are normally distributed, then the extended Kalman filter can be applied to obtain an estimate of the state vector x_k given the observations y_k at time step k.

In the extended Kalman filter algorithm, the states in the prediction step are predicted as

$$\hat{x}_{k|\,k-1} = f_k(\hat{x}_{k-1}, 0)$$

The non-linear functions in the state transition and measurement equations are linearized to obtain the covariance using Jacobian matrices:

$$A_{ij} = \frac{\partial f_i(\hat{x}_{k-1}, 0)}{\partial x_j}, \qquad W_{ij} = \frac{\partial f_i(\hat{x}_{k-1}, 0)}{\partial w_j}$$

$$H_{ij} = \frac{\partial h_i\left(\hat{x}_{k|\,k-1}, 0\right)}{\partial x_j}, \qquad U_{ij} = \frac{\partial f_i\left(\hat{x}_{k|\,k-1}, 0\right)}{\partial v_j}$$

The predicted state covariance is thus

$$P_k^- = A_k P_{k-1} A_k^T + W_k Q_{k-1} W_k^T$$

Prediction of the measurement is given by

$$\hat{y}_k = h_k\left(\hat{x}_{k|\,k-1}, 0\right)$$

with covariance

$$F_k = H_k P_k^- H_k^T + U_k R_k U_k^T$$

The state vector x_k is estimated using the predicted states $\hat{x}_{k|\,k-1}$ in the update step. The measurement residual is weighted by the Kalman gain K_k,

$$\hat{x}_{k|\,k} = \hat{x}_{k|\,k-1} + K_k b_k$$

where the measurement residual is $b_k = y_k - \hat{y}_k$.

The optimal gain is

$$K_k = P_k^- H_k^T (F_k)^{-1}$$

and the updated covariance

$$P_k = P_k^- - K_k H_k P_k^-$$

2.3 Unscented Kalman Filter

A weakness of the extended Kalman filter is that it provides poor estimates for systems that are highly non-linear. The Kalman gain is obtained from the covariance of the measurement and the states. Thus a poor estimate of the covariance leads to unreliable values for the Kalman gain.

The unscented Kalman filter was proposed by Julier and Uhlmann (1996). They showed that for systems that are highly non-linear with normal distributions, the unscented Kalman filter estimates more accurately as compared to the extended Kalman filter. In addition, they also demonstrated the difficulty in implementing the extended Kalam filter since approximate techniques for the computation of Jacobian matrices are required.

The unscented Kalman filter makes an approximation of the state random variable distribution using a set of chosen deterministic sample points known as sigma points which capture the states mean and the states covariance. Two weights are assigned to each sigma point.

An example is given as follows

Let

$$y = q(x), \qquad (2.3.1)$$

q represents a function that is non-linear. Given the pdf of x which is Gaussian, the pdf of y is obtained as given below:

Let $\dim(x) = L$, $\mathrm{E}[x] = \hat{x}$ and the covariance matrix of x is P_x. A set of $2L+1$ weighted sigma points $\{W_i, X(i)\}$ are generated with

$$X(0) = \hat{x}$$

$$X(i = 1, \ldots\ldots, L) = \hat{x} + \left(\sqrt{(L+\lambda)P_x}\right)_i$$

$$X(i = L+1, \ldots\ldots, 2L) = \hat{x} - \left(\sqrt{(L+\lambda)P_x}\right)_{i-L}$$

A scaling parameter λ is defined as

$$\lambda = \alpha^2(L+\kappa) - L$$

and α determines the spreads of sigma points around x, and usually set as small as possible. κ is a secondary scaling parameter.

A set of two weights is assigned to each sigma point $X(i)$ for $i = 1, \ldots\ldots, 2L$

$$W_0^{(m)} = \frac{\lambda}{L+\lambda} \qquad W_0^{(m)} = \frac{\lambda}{L+\lambda} + 1 - \alpha^2 + \beta$$

$$W_i^{(m)} = \frac{1}{2(L+\lambda)} \qquad W_i^{(c)} = \frac{1}{2(L+\lambda)} \qquad (2.3.2)$$

β contains prior information of the distribution of x. An optimal value of $\beta = 2$ is used for normal distributions.

We then proceed as follows:

$$Y(i = L + 1, \dots \dots, 2L) = q\big(X(i)\big)$$

Then the mean of y is given by

$$\hat{y} = \sum_{i=1}^{2L} W_i^{(m)} Y(i)$$

with covariance

$$P = \sum_{i=1}^{2L} W_i^{(c)} (Y(i) - \hat{y})(Y(i) - \hat{y})^T$$

Wan and Van Der Merwe (2000) show how to obtain the unscented Kalman filter algorithm. Given that we have a non-linear state transition Equation 1.0.1 and a non-linear measurement Equation 1.0.2. Let the dimensions of the state noise and the state process be:

$$N_x = \dim(x), \qquad N_w = \dim(w)$$

and dimensions of the measurement noise and measurement process be:

$$N_y = \dim(y), \qquad N_v = \dim(v)$$

respectively.

Let $L = N_x + N_w + N_v$, and:

$$x_k^a = [x_k, w_k, v_k]^T \tag{2.3.3}$$

(x_k^a) is an L-dimensional column vector whose entries are the state process, and the state and measurement noise. An assumption is made that (x_k^a) has a mean $\bar{x}$ and covariance matrix P_x.

Let

$$\chi^a = [\chi_0, \ \chi_1, \dots \dots, \chi_{2L}]$$

χ^a is an $L \times (2L + 1)$ –matrix of sigma points, with columns defined by

$$\chi_0 = \bar{x}$$

$$\chi_i = \bar{x} + \left(\sqrt{(L + \lambda)P_x}\right)_i \qquad for\ i = 1, \dots \dots, L$$

$$\chi_i = \bar{x} - \left(\sqrt{(L + \lambda)P_x}\right)_{i-L} \qquad for\ i = L + 1, \dots \dots, 2L$$

The matrix χ^a can be decomposed as follows:

$$\chi^a = \begin{bmatrix} \chi^x \\ \chi^w \\ \chi^v \end{bmatrix}$$

χ^x is $N_x \times (2L + 1)$-dimensional,

χ^w is $N_w \times (2L + 1)$-dimensional,

χ^v is $N_v \times (2L + 1)$-dimensional.

2.4 Particle Filter

It's also known as the Sequential Monte Carlo method. It can be used to approximate filter distributions in state space models. The particle filter is an optimal method for a non-linear system that is non-Gaussian. Both the unscented and extended Kalman filters cannot be used as optimal methods when a non-linear system is non-Gaussian.

The particle filter method makes use of Monte Carlo simulation to obtain the posterior density function in Equation 1.0.3. Numerous iterations are then performed to approximate the state distributions based on the

random samples obtained from the set of random numbers used in the Monte Carlo simulation. The key idea is to represent the required posterior density function by a set of random samples known as samples with associated weights and to compute the estimates based on these samples and weights.

The state distributions are then approximated with a finite set of weighted random samples drawn from a known, proposal distribution $\left(q(x_{0:k}|\, y_{1:k})\right)$. These random samples are called particles and at time step k we might denote n particles for the state x_k as

$$x_k^{(1)}, x_k^{(2)}, \dots\dots, x_k^{(n)}.$$

The particles $x_{0:k}^{(i)}$ for $i = 1, \dots\dots, n$ are independent and identically distributed.

Each particle is assigned an importance weight (r_k) which determines its probability of being sampled from the proposal distribution. The weighted set of n particles at time step k will be denoted as $\{x_k^{(i)}, r_k^{(i)}\}$ for all $i = 1, \dots\dots, n$.

The posterior density function from Equation 1.0.3 can be approximated as follows

$$p(x_{0:k}|\, y_{1:k})$$

$$\coloneqq \sum_{i=1}^{n} r_k^{(i)} \delta\left(x_{0:k} - x_{0:k}^{(i)}\right) \qquad (2.4.1)$$

where $\delta(.)$ is a delta function. All we need to do to evaluate the transition probability in Equation 2.4.1, we need to generate a set of particles from a proposal distribution and iteratively compute the importance weights. This is grouped into three steps: sampling, computing the particle weights and resampling.

3. Stochastic Volatility Models

After the October 1987 stock market crash, significant variations from normality have shown up in the term structure of volatility. Various academicians and traders have taken a keen interest on this observation and as a result, a lot of work has been done on this area. The danger of models used for pricing based on an incorrect assumption of log-normality is the risk of obtaining biased prices.

The Black-Scholes model which has been used extensively in the past is considered to be successful in asset pricing both in terms of approach and applicability. With the assumption of geometric brownian motion, the risk neutral density for the underlying assets is taken to be lognormal. Asset prices are often observed to have random volatility. These observations cannot be accurately be assumed to have a lognormal density since the density functions are fat-tailed and skewed. Stochastic volatility models are widely used in the finance industry for derivative pricing and hedging. They are popular is their flexibility in capturing the volatility surface.

The Heston model makes the assumption of stochastic volatility in the pricing of European call option and obtains a closed-form solution. The model further assumes that the volatility and the underlying asset price are correlated. In so doing, the Heston model is enable to capture various properties of the financial information which the Black-Scholes model doesn't.

The double Heston model was proposed by Christofiersen et al. (2009) to deal with the failure of the standard Heston Model to not always capture the term structure dynamics of the implied volatility especially in cases where the maturity period is short. In the Double Heston model, an asset return is driven by two-factor stochastic volatility. This has the advantage of improving the model's flexibility in modelling the volatility term structure.

In this chapter, we describe the Standard Heston model and its extension, the Double Heston model, in detail and present their characteristic functions, which are important in option valuations. We then extend the Double Heston model to the case where we have two underlying assets. We also present the state-space representations for these models, which we use in the filtering methods to estimate the volatilities.

3.1 The Heston Model

In this section, we first present the dynamic system for the Heston model under a risk-neutral measure $\mathbb{Q}$. Under a risk-neutral measure $\mathbb{Q}$, the Heston (1993) model assumes that an underlying stock price, S_t has a stochastic variance, V_t, that follows a Cox, Ingersoll and Ross (1985) process. This process is represented by the following dynamical system:

$$dS_t = (r - q)S_t dt + \sqrt{V_t} S_t dW_t$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dZ_t$$

where r is a constant risk-free interest rate, q is a constant dividend. The terms κ, θ and V_t above describe the mean-reverting volatility of the process. The mean speed of reversion, κ, determines the relative speed of the volatility or the weight that the long-run variance and current variance are given. The average level of the stock, θ, is the long-run variance that the drift pulls the volatility towards. The V_t term is the current variance, while σ is the volatility of the volatility. All the parameters κ, θ and σ are positive constant. The terms W_t and Z_t are Wiener processes that must be correlated with each other, that is;

$$(dW_t dZ_t) = \rho dt$$

In the above equation the term ρ is the correlation coefficient between the return of the underlying asset and the changes in the variance. This correlation has proven to be a great advantage to the Heston model as this is also present in empirical studies that have been performed over the years. The correlation, which is often negative, will ensure that the volatility for example will rise if the underlying asset value falls dramatically. In addition the variance is also mean-reverting, which is also evident in the market. The mean-reverting process is the term $\kappa(\theta - v)$.

For option valuation, we follow the Albrecher et al. (2006) approach, such that the characteristic function of log returns $x_k = ln(S_k/S_{k-1})$ $(for\ k \leq t)$ of the Heston model is derived using the so called the little Heston trap. This characteristic function is only slightly different from the original formulation of Heston (1993), but it provides a better computation of the numerical integration. The European call option price under the basic Heston model in the one dimensional framework is given by;

$$C(S, V, K, \tau) = S_k e^{-q\tau} P_1 - K e^{-r\tau} P_2 \qquad (3.1.1)$$

Where $P_j (j = 1,2)$ are the risk-adjusted probabilities of the log of the underlying price $x_t = \ln(S_t/(S_{t-1})$. K denotes the strike price.

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\emptyset \ln K} f_j(\emptyset; x_k, V_k)}{i\emptyset}\right] d\emptyset$$

for $j = 1, 2$.

The characteristic functions $f_j(\emptyset; x_k, V_k)$ in the probabilities are given by

$$f_j(\emptyset; x_k, V_k) = e^{i\emptyset \ln K + A_j(\emptyset,\tau) + B_j(\emptyset,\tau)V_k}$$

Where

$$B_j(\emptyset, \tau) = \frac{b_j - \rho\sigma\phi i + d_j}{\sigma^2}\left[\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right],$$

$$A_j(\emptyset, \tau) = r\phi i\tau + \frac{a}{\sigma^2}\left[(b_j - \rho\sigma\phi i + d_j)\tau - 2\, ln\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right)\right],$$

$$g_j = \frac{b_j - \rho\sigma\phi i + d_j}{b_j - \rho\sigma\phi i + d_j},$$

$$d_j = \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2(2u_j\phi i - \phi^2)}$$

And $i = \sqrt{-1}$, $\tau = T - k$, $u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$, $a = \kappa\theta$, $b_1 = \kappa - \rho\sigma$, $b_2 = \kappa$ and ϕ is called the integration variable or node.

3.2 The Double Heston Model

The model makes an assumption that the underlying stock price, S_t is driven by two independent factors of volatility, V_t^1 and V_t^2. The dynamics of the system are given as follows assuming a risk neutral framework:

$$dS_t = (r-q)S_t dt + \sqrt{V_t^1} S_t dW_t^1 + \sqrt{V_t^2} S_t dW_t^2$$

$$dV_t^1 = \kappa_1(\theta_1 - V_t^1)dt + \sigma_1\sqrt{V_t^1}dZ_t^1$$

$$dV_t^2 = \kappa_2(\theta_2 - V_t^2)dt + \sigma_2\sqrt{V_t^2}dZ_t^2$$

(3.2.1)

where r is the deterministic risk-free rate of interest,

q denotes the dividend-yield.

An assumption is made that all the parameters are constant.

The Brownian motions W_t^1, Z_t^1 and W_t^2, Z_t^2 are correlated.

$$d[W_t^i, Z_t^j] = \rho_i dt \qquad for\ all \quad i = j$$
$$d[W^i, Z^j]_t = 0 \qquad for\ all \quad i \neq j$$

for $i, j = 1, 2$.

The multidimensional Feynman-Kac Theorem is used to obtain the characteristic function for the Double Heston model

Theorem 3.1. Multi-dimensional Feynman-Kac Theorem

Let x_k be an n-dimensional stochastic process with dynamics

$$dx_k = \mu(k, x_k)dk + \sigma(k, x_k)dW_k$$

(3.2.2)

Where $k \leq t \leq T$.

The infinitesimal generator of the process in Equation 3.1.2 is defined by

$$\mathcal{A} = \sum_{i=1}^{n} \mu_i(k, x^1, \dots, x^n)\frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i=1}^{n} C_{ij}\frac{\partial^2}{\partial x^i \partial x^j}$$

(3.2.3)

where $C_{ij} = (\sigma\sigma^T)_{ij}$.

Theorem 3.1 implies that f satisifies the PDF:

$$\frac{\partial f}{\partial t} + \mathcal{A}f - rf = 0$$

(3.2.4)

Applying Ito's lemma and Equation 3.2.1, returns $x_k = \ln(\frac{S_k}{S_{k-1}})$ are given by

$$dx_k = \left(r - q - \frac{1}{2}(V_k^1 + V_k^2)\right)dk + \sqrt{V_k^1}dW_k^1 + \sqrt{V_k^2}dW_k^2$$

Thus, the dynamical system of the Double Heston model can be given as:

$$\begin{pmatrix} dx_k \\ dV_k^1 \\ dV_k^2 \end{pmatrix} = \begin{pmatrix} \left(r - q - \frac{1}{2}(V_k^1 + V_k^2)\right)dk + \sqrt{V_k^1}dW_k^1 + \sqrt{V_k^2}dW_k^2 \\ \kappa_1(\theta_1 - V_k^1)dk + \sigma_1\sqrt{V_k^1}dZ_k^1 \\ \kappa_2(\theta_2 - V_k^2)dk + \sigma_2\sqrt{V_k^2}dZ_k^2 \end{pmatrix}$$

If we set

$$Z_1 = \rho_1 W_1 + \sqrt{1-\rho_1^2}\, W_3$$

$$Z_2 = \rho_2 W_2 + \sqrt{1-\rho_2^2}\, W_4$$

where W_1, W_2, W_3, W_4 are independent Brownian motions. Then the volatility matrix from Theorem 3.1 is given by

$$\sigma(x_k, k) = \begin{pmatrix} \sqrt{V_k^1} & \sqrt{V_k^2} & 0 & 0 \\ \sigma_1\sqrt{V_k^1}\rho_1 & 0 & \sigma_1\sqrt{V_k^1(1-\rho_1^2)} & 0 \\ 0 & \sigma_2\sqrt{V_k^2}\rho_2 & 0 & \sigma_2\sqrt{V_k^2(1-\rho_2^2)} \end{pmatrix}$$

So that

$$\sigma\sigma^T = \begin{pmatrix} V_k^1 + V_k^2 & \sigma_1 V_k^1 \rho_1 & \sigma_2 V_k^2 \rho_2 \\ \sigma_1 V_k^1 \rho_1 & \sigma_1^2\, V_k^1 & 0 \\ \sigma_2 V_k^2 \rho_2 & 0 & \sigma_2^2\, V_k^2 \end{pmatrix}$$

and the drift is given by

$$\mu = \begin{pmatrix} r - q - \frac{1}{2}(V_k^1 + V_k^2) \\ \kappa_1(\theta_1 - V_k^1) \\ \kappa_2(\theta_2 - V_k^2) \end{pmatrix}$$

Then the generator $\mathcal{A}$ as given in Equation 3.2.3 becomes

$$\mathcal{A} = \left(r - q - \frac{1}{2}(V_k^1 + V_k^2)\right)\frac{\partial f}{\partial x_k} + \kappa_1(\theta_1 - V_k^1)\frac{\partial f}{\partial V_k^1} + \kappa_2(\theta_2 - V_k^2)\frac{\partial f}{\partial V_k^2} + \frac{1}{2}(V_k^1 + V_k^2)\frac{\partial^2 f}{\partial x_k^2} + \rho_1\sigma_1 V_k^1 \frac{\partial^2 f}{\partial x_k V_k^1}$$
$$+ \rho_2\sigma_2 V_k^2 \frac{\partial^2 f}{\partial x_k V_k^2} + \frac{1}{2}\sigma_1^2 V_k^1 \frac{\partial^2 f}{\partial V_k^{1\,2}} + \frac{1}{2}\sigma_2^2 V_k^2 \frac{\partial^2 f}{\partial V_k^{2\,2}}. \tag{3.2.5}$$

The double Heston model Partial Differential Equation is obtained by substituting $\mathcal{A}$ into Equation 3.2.4.

According to Christoffersen et al. (2009), the Double Heston model belongs to the class of affine models. This means that f has a closed-form solution with an exponential affine relationship to the state variables which can take the form that follows:

$$f(\emptyset_0, \emptyset_1, \emptyset_2;\, x_k, V_k^1, V_k^2) = \mathbb{E}[exp(i\emptyset_0 x_T + i\emptyset_1 V_T^1 + i\emptyset_2 V_T^2)]$$
$$= exp(A(\tau) + B_0(\tau)x_k + B_1(\tau)V_k^1 + + B_2(\tau)V_k^2) \tag{3.2.6}$$

Where $\tau = T - k$

The coefficients A, B_0, B_1, B_2 can be obtained as follows. We first substitute Equation 3.2.6 for f in Equation 3.2.4 to obtain

$$f\left[\left(\frac{\partial A}{\partial k} + \frac{\partial B_0 x_k}{\partial k} + \frac{\partial B_1 V_k^1}{\partial k} + \frac{\partial B_2 V_k^2}{\partial k}\right) + \mu_1 B_0 + \mu_2 B_1 + \mu_3 B_2\right.$$
$$\left. + \frac{1}{2}((\sigma\sigma^T)_{11}B_0^2 + (\sigma\sigma^T)_{22}B_1^2 + (\sigma\sigma^T)_{33}B_2^2 + (\sigma\sigma^T)_{12}B_0B_1 + (\sigma\sigma^T)_{13}B_0B_2)\right]$$
$$= 0 \tag{3.2.7}$$

Note that μ and $\sigma\sigma^T$ are affine, such that

$$\mu(x_k) = K_0 + K_1 x_k + K_2 V_k^1 + K_3 V_k^2$$
$$\sigma(x_k)\sigma(x_k)^T = H_0 + H_1 x_k + H_2 V_k^1 + H_3 V_k^2$$

Where

$$K_0 = \begin{pmatrix} r-q \\ \kappa_1\theta_1 \\ \kappa_2\theta_2 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} -\frac{1}{2} \\ -\kappa_1 \\ 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ -\kappa_2 \end{pmatrix}$$

And

$$H_0 = H_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & \sigma_1\rho_1 & 0 \\ \sigma_1\rho_1 & \sigma_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 & \sigma_2\rho_2 \\ 0 & 0 & 0 \\ \sigma_2\rho_2 & 0 & \sigma_2^2 \end{pmatrix}$$

Substituting the variables from μ and $\sigma\sigma^T$ in the Equation 3.1.5, we get

$$f\left[\left(\frac{\partial A}{\partial k} + \frac{\partial B_0 x_k}{\partial k}\right) + (r-q)B_0 + \kappa_1\theta_1 B_1 + \kappa_2\theta_2 B_2 + \frac{\partial B_1 V_k^1}{\partial k}\right.$$
$$+ V_k^1\left(-\frac{1}{2}B_0 - \kappa_1 B_1 + \frac{1}{2}B_0^2 + \frac{1}{2}\sigma_1^2 B_1^2 + \frac{1}{2}\sigma_1\rho_1 B_0 B_1\right) + \frac{\partial B_2 V_k^2}{\partial k}$$
$$\left. + V_k^2\left(-\frac{1}{2}B_0 - \kappa_2 B_2 + \frac{1}{2}B_0^2 + \frac{1}{2}\sigma_2^2 B_2^2 + \frac{1}{2}\sigma_2\rho_2 B_0 B_2\right)\right] = 0$$

We will drop f because it is always true that $f > 0$. In order for the drift term to equal 0 for all values of x_k, V_k^1 and V_k^2, their coefficient terms and the constants terms must sum to 0. That gives us the following system of ODEs

$$\frac{\partial B_0}{\partial k} = 0$$
$$\frac{\partial A}{\partial k} + (r-q)B_0 + \kappa_1\theta_1 B_1 + \kappa_2\theta_2 B_2 = 0$$
$$\frac{\partial B_1 V_k^1}{\partial k} - \frac{1}{2}B_0 - \kappa_1 B_1 + \frac{1}{2}B_0^2 + \frac{1}{2}\sigma_1^2 B_1^2 + \frac{1}{2}\sigma_1\rho_1 B_0 B_1 = 0$$
$$\frac{\partial B_2}{\partial k} - \frac{1}{2}B_0 - \kappa_2 B_2 + \frac{1}{2}B_0^2 + \frac{1}{2}\sigma_2^2 B_2^2 + \frac{1}{2}\sigma_2\rho_2 B_0 B_2 = 0$$

(3.2.8)

These are known as Riccati equations. Rouah (2013) provided solutions to the Riccati equations for the Standard Heston model equations. Rouah (2013) also argue that B_1 and B_2 are identical to their counterparts in the Standard Heston model, therefore their solutions are

$$B_0(\tau) = 0$$
$$B_j(\tau, \phi) = \frac{\kappa_j - \rho_j\sigma_j\phi i + d_j}{\sigma_j^2}\left[\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right]$$
$$A(\tau, \phi) = (r-q)\phi i\tau + \sum_{j=1}^{2}\frac{\kappa_j\theta_j}{\sigma_j^2}\left[(\kappa_j - \rho_j\sigma_j\phi i + d_j)\tau - 2ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right]$$

(3.2.9)

where

$$g_j = \frac{\kappa_j - \rho_j\sigma_j\phi i + d_j}{\kappa_j - \rho_j\sigma_j\phi i + d_j}$$

$$d_j = \sqrt{\left(\kappa_j - \rho_j \sigma_j \phi i\right)^2 + \sigma_j^2 \phi(\phi + i)}$$

For $j = 1,2$.

With the known coefficients A, B_0, B_1, B_2, the characteristic function f can now be obtained. Using Fourier inversion, Christofiersen et al. (2009) obtained the price of a European call option under the Double Heston model as:

$$C(K) = S_k e^{-q\tau} P_1 - K e^{-r\tau} P_2$$

K denotes the exercise price,

$$P_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left[\frac{e^{-i\phi \ln K} f(\phi - i;\ x_k, V_k^1, V_k^2)}{i\phi S_t e^{-\tau}}\right] d\phi$$

$$P_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left[\frac{e^{-i\phi \ln K} f(\phi;\ x_k, V_k^1, V_k^2)}{i\phi}\right] d\phi$$

3.3 The Double Heston Model with Two Underlying Assets

We now extend the Double Heston model discussed above to the case where we have two underlying assets. We assume that we have two underlying stock prices, S_t^1 and S_t^2 is driven by two independent factors of volatility, V_t^1 and V_t^2 respectively. The dynamics of the system are given as follows assuming a risk neutral framework:

$$dS_t^1 = (r_1 - q_1) S_t^1 dt + \sqrt{V_t^1 S_t^1 dW_t^1} + S_t^1 a_1 dQ_t^1$$

$$dS_t^2 = (r_2 - q_2) S_t^2 dt + \sqrt{V_t^2 S_t^2 dW_t^2} + S_t^2 a_2 dQ_t^2$$

$$dV_t^1 = \kappa_1(\theta_1 - V_t^1)dt + \sigma_1 \sqrt{V_t^1} dZ_t^1$$

$$dV_t^2 = \kappa_2(\theta_2 - V_t^2)dt + \sigma_2 \sqrt{V_t^2} dZ_t^2 \qquad (3.3.1)$$

where $r_i, q_i, a_i, \kappa_i, \theta_i, \sigma_i$ for $i = 1,2$ are known constants, such that:

$$a_i, \kappa_i, \theta_i, \sigma_i > 0$$

$$\frac{2\,\kappa_i \theta_i}{\varepsilon_i^2} > 1,$$

and W_t^i, Q_t^i, Z_t^i are standard Weiner processes, For $i = 1,2$, we make the assumption that:

$$d[W^i, Q^j] = 0 \qquad for\ all \quad i,j$$

$$d[Z^i, Q^j] = 0 \qquad for\ all \quad i,j$$

$$d[W^i, Z^j] = 0 \qquad for\ all \quad i \neq j$$

$$d[W^i, W^j] = 0 \qquad for\ all \quad i \neq j$$

$$d[Z^i, Z^j] = 0 \qquad for\ all \quad i \neq j$$

$$d[Q^i, Q^j] = \rho_{i,j} dt \qquad for\ all \quad i \neq j$$

$$d[W^i, Z^j] = \rho_i dt \qquad for\ all \quad i = j$$

$$d[W^i, W^j] = dt \qquad for\ all \quad i = j$$

$$d[Z^i, Z^j] = dt \qquad for\ all \quad i = j$$

$$d[Q^i, Q^j] = dt \qquad for\ all \quad i = j$$

$for\ i, j = 1,2.$

In this study, we consider the case where the two assets are uncorrelated, that is $a_i = 0$, for $i = 1,2$. The model thus becomes;

$$dS_t^1 = (r_1 - q_1)S_t^1 dt + \sqrt{V_t^1} S_t^1 dW_t^1$$

$$dS_t^2 = (r_2 - q_2)S_t^2 dt + \sqrt{V_t^2} S_t^2 dW_t^2$$

$$dV_t^1 = \kappa_1(\theta_1 - V_t^1)dt + \sigma_1\sqrt{V_t^1} dZ_t^1$$

$$dV_t^2 = \kappa_2(\theta_2 - V_t^2)dt + \sigma_2\sqrt{V_t^2} dZ_t^2 \qquad (3.3.2)$$

The multidimensional Feynman-Kac Theorem is used to obtain the characteristic function for the Double Heston model written on two underlying assets as follows:

Let $x_k^i\ for\ i = 1,2$ be an n-dimensional stochastic process with dynamics

$$dx_k^i = \mu_i(k, x_k^i)dk + \sigma(k, x_k^i)dW_k^i \qquad (3.3.3)$$

Where $k \le t \le T$.

The infinitesimal generator of the process in Equation 3.3.3 is defined by

$$\mathcal{A} = \sum_{i=1}^{n} \mu_i(k, x^1, \dots, x^n)\frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i=1}^{n} C_{ij}\frac{\partial^2}{\partial x^i \partial x^j}$$

Where $C_{ij} = (\sigma\sigma^T)_{ij}$.

Applying Ito's lemma, returns $x_k^i = \ln(\frac{S_k^i}{S_{k-1}^i})$ for $i = 1,2$ are given by

$$dx_k^1 = \left(r_1 - q_1 - \frac{1}{2}(V_k^1)\right)dk + \sqrt{V_k^1} dW_k^1 + a_1 dQ_t^1$$

$$dx_k^2 = \left(r_2 - q_2 - \frac{1}{2}(V_k^2)\right)dk + \sqrt{V_k^2} dW_k^2 + a_2 dQ_t^2$$

Thus, the dynamical system of the Double Heston model written on two underlying assets can be given as:

$$\begin{pmatrix} dx_k^1 \\ dx_k^2 \\ dV_k^1 \\ dV_k^2 \end{pmatrix} = \begin{pmatrix} \left(r_1 - q_1 - \frac{1}{2}(V_k^1)\right)dk + \sqrt{V_k^1} dW_k^1 + a_1 dQ_t^1 \\ \left(r_2 - q_2 - \frac{1}{2}(V_k^2)\right)dk + \sqrt{V_k^2} dW_k^2 + a_2 dQ_t^2 \\ \kappa_1(\theta_1 - V_k^1)dk + \sigma_1\sqrt{V_k^1} dZ_k^1 \\ \kappa_2(\theta_2 - V_k^2)dk + \sigma_2\sqrt{V_k^2} dZ_k^2 \end{pmatrix}$$

If we set

$$Z_1 = \rho_1 W_1 + \sqrt{1 - \rho_1^2}\, W_3$$

$$Z_2 = \rho_2 W_2 + \sqrt{1 - \rho_2^2}\, W_4$$

where W_1, W_2, W_3, W_4 are independent Brownian motions. Then the volatility matrix from Theorem 3.1 is given by

$$\sigma(x_k,k)=\begin{pmatrix}\sqrt{V_k^1} & 0 & 0 & 0\\ 0 & \sqrt{V_k^2} & 0 & 0\\ \sigma_1\sqrt{V_k^1}\rho_1 & 0 & \sigma_1\sqrt{V_k^1(1-\rho_1^2)} & 0\\ 0 & \sigma_2\sqrt{V_k^2}\rho_2 & 0 & \sigma_2\sqrt{V_k^2(1-\rho_2^2)}\end{pmatrix}$$

So that

$$\sigma\sigma^T=\begin{pmatrix}V_k^1 & 0 & \sigma_1V_k^1\rho_1 & 0\\ 0 & V_k^2 & 0 & \sigma_2V_k^2\rho_2\\ \sigma_1V_k^1\rho_1 & 0 & \sigma_1^2\ V_k^1 & 0\\ 0 & \sigma_2V_k^2\rho_2 & 0 & \sigma_2^2\ V_k^2\end{pmatrix}$$

and the drift is given by

$$\mu=\begin{pmatrix}r_1-q_1-\frac{1}{2}(V_k^1)\\ r_2-q_2-\frac{1}{2}(V_k^2)\\ \kappa_1(\theta_1-V_k^1)\\ \kappa_2(\theta_2-V_k^2)\end{pmatrix}$$

Then the generator $\mathcal{A}$ as given in Equation 3.2.3 becomes

$$\mathcal{A}=\left(r_1-q_1-\frac{1}{2}(V_k^1)\right)\frac{\partial f}{\partial x_k^1}+\left(r_2-q_2-\frac{1}{2}(V_k^2)\right)\frac{\partial f}{\partial x_k^2}+\kappa_1(\theta_1-V_k^1)\frac{\partial f}{\partial V_k^1}+\kappa_2(\theta_2-V_k^2)\frac{\partial f}{\partial V_k^2}+\frac{1}{2}V_k^1\frac{\partial^2 f}{\partial x_k^{1\,2}}$$
$$+\frac{1}{2}V_k^2\frac{\partial^2 f}{\partial x_k^{2\,2}}+\rho_1\sigma_1V_k^1\frac{\partial^2 f}{\partial x_k^1V_k^1}+\rho_2\sigma_2V_k^2\frac{\partial^2 f}{\partial x_k^2V_k^2}+\frac{1}{2}\sigma_1^2V_k^1\frac{\partial^2 f}{\partial V_k^{1\,2}}+\frac{1}{2}\sigma_2^2V_k^2\frac{\partial^2 f}{\partial V_k^{2\,2}} \quad (3.3.4)$$

The double Heston model written on two underlying assets Partial Differential Equation is obtained by substituting $\mathcal{A}$ into Equation 3.2.4.

Following a similar approach to Christoffersen et al. (2009), the Double Heston model written on two underlying assets belongs to the class of affine models. This means that f has a closed-form solution with an exponential affine relationship to the state variables which can take the form that follows:

$$f(\emptyset_0,\emptyset_1,\emptyset_2,\emptyset_3;\ x_k^1,x_k^2,V_k^1,V_k^2)=\mathbb{E}[exp(i\emptyset_0x_T^1+i\emptyset_1x_T^2+i\emptyset_2V_T^1+i\emptyset_3V_T^2)]$$
$$=exp(A(\tau)+B_0(\tau)x_k^1+B_1(\tau)x_k^2+B_2(\tau)V_k^1++B_3(\tau)V_k^2) \quad (3.3.5)$$

Where $\tau=T-k$

The coefficients A,B_0,B_1,B_2,B_3 can be obtained as follows.

$$f\left[\left(\frac{\partial A}{\partial k}+\frac{\partial B_0x_k^1}{\partial k}+\frac{\partial B_1x_k^2}{\partial k}+\frac{\partial B_2V_k^1}{\partial k}+\frac{\partial B_3V_k^2}{\partial k}\right)+\mu_1B_0+\mu_2B_1+\mu_3B_2+\mu_4B_3\right.$$
$$\left.+\frac{1}{2}((\sigma\sigma^T)_{11}B_0^2+(\sigma\sigma^T)_{22}B_1^2+(\sigma\sigma^T)_{33}B_2^2+(\sigma\sigma^T)_{13}B_0B_2+(\sigma\sigma^T)_{24}B_1B_2+(\sigma\sigma^T)_{44}B_3^2)\right]$$
$$=0 \quad (3.3.6)$$

Note that μ and $\sigma\sigma^T$ are affine, such that

$$\mu(x_k)=K_0+K_1x_k^1+K_2x_k^2+K_3V_k^1+K_4V_k^2$$
$$\sigma(x_k)\sigma(x_k)^T=H_0+H_1x_k^1+H_2x_k^2+H_2V_k^1+H_3V_k^2$$

Where

$$K_0 = \begin{pmatrix} r_1 - q_1 \\ r_2 - q_2 \\ \kappa_1\theta_1 \\ \kappa_2\theta_2 \end{pmatrix}, \quad K_1 = K_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ -\kappa_1 \\ 0 \end{pmatrix}, \quad K_4 = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ -\kappa_2 \end{pmatrix}$$

And

$$H_0 = H_1 = H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 1 & 0 & \sigma_1\rho_1 & 0 \\ 0 & 0 & 0 & 0 \\ \sigma_1\rho_1 & 0 & \sigma_1^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$H_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \sigma_2\rho_2 \\ 0 & 0 & 0 & 0 \\ 0 & \sigma_2\rho_2 & 0 & \sigma_2^2 \end{pmatrix}$$

Substituting the variables from μ and $\sigma\sigma^T$ in the Equation 3.3.5, we get

$$f\left[\left(\frac{\partial A}{\partial k} + \frac{\partial B_0 x_k^1}{\partial k} + \frac{\partial B_1 x_k^2}{\partial k}\right) + (r_1 - q_1)B_0 + (r_2 - q_2)B_1 + \kappa_1\theta_1 B_2 + \kappa_2\theta_2 B_3 + \frac{\partial B_2 V_k^1}{\partial k}\right.$$
$$+ V_k^1\left(-\frac{1}{2}B_0 - \kappa_1 B_2 + \frac{1}{2}B_0^2 + \frac{1}{2}\sigma_1^2 B_2^2 + \frac{1}{2}\sigma_1\rho_1 B_0 B_2\right) + \frac{\partial B_3 V_k^2}{\partial k}$$
$$\left. + V_k^2\left(-\frac{1}{2}B_1 - \kappa_2 B_3 + \frac{1}{2}B_1^2 + \frac{1}{2}\sigma_2^2 B_3^2 + \frac{1}{2}\sigma_2\rho_2 B_1 B_3\right)\right] = 0 \tag{3.3.7}$$

We will drop f because it is always true that $f > 0$. In order for the drift term to equal 0 for all values of x_k^1, x_k^2, V_k^1 and V_k^2, their coefficient terms and the constants terms must sum to 0. That gives us the following system of ODEs

$$\frac{\partial B_0}{\partial k} + \frac{\partial B_1}{\partial k} = 0$$
$$\frac{\partial A}{\partial k} + (r_1 - q_1)B_0 + (r_2 - q_2)B_1 + \kappa_1\theta_1 B_2 + \kappa_2\theta_2 B_3 = 0$$
$$\frac{\partial B_2}{\partial k} - \frac{1}{2}B_0 - \kappa_1 B_2 + \frac{1}{2}B_0^2 + \frac{1}{2}\sigma_1^2 B_2^2 + \frac{1}{2}\sigma_1\rho_1 B_0 B_2 = 0$$
$$\frac{\partial B_3}{\partial k} - \frac{1}{2}B_1 - \kappa_2 B_3 + \frac{1}{2}B_1^2 + \frac{1}{2}\sigma_2^2 B_3^2 + \frac{1}{2}\sigma_2\rho_2 B_1 B_3 = 0 \tag{3.3.8}$$

These are known as Riccati equations. Rouah (2013) provided solutions to the Riccati equations for the Standard Heston model equations. Rouah (2013) also argue that B_2 and B_3 are identical to their counterparts in the Standard Heston model, therefore their solutions are

$$B_0(\tau) = B_1(\tau) = 0$$
$$B_j(\tau, \phi) = \frac{\kappa_j - \rho_j\sigma_j\phi i + d_j}{\sigma_j^2}\left[\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right]$$
$$A(\tau, \phi) = (r - q)\phi i\tau + \sum_{j=1}^{2}\frac{\kappa_j\theta_j}{\sigma_j^2}\left[(\kappa_j - \rho_j\sigma_j\phi i + d_j)\tau - 2ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right] \tag{3.3.9}$$

Where

$$g_j = \frac{\kappa_j - \rho_j \sigma_j \phi i + d_j}{\kappa_j - \rho_j \sigma_j \phi i + d_j}$$

$$d_j = \sqrt{\left(\kappa_j - \rho_j \sigma_j \phi i\right)^2 + \sigma_j^2 \phi(\phi + i)}$$

For $j = 2,3$.

With the known coefficients A, B_0, B_1, B_2, the characteristic function f can now be obtained. Using Fourier inversion, Christofiersen et al. (2009) obtained the price of a European call option under the Double Heston model with two underlying assets as:

$$C(K) = S_k e^{-q\tau} P_1 - K e^{-r\tau} P_2$$

K denotes the exercise price,

$$P_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left[\frac{e^{-i\phi \ln K} f(\phi - i;\ x_k^1, x_k^2, V_k^1, V_k^2)}{i\phi S_t e^{-\tau}}\right] d\phi$$

$$P_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left[\frac{e^{-i\phi \ln K} f(\phi;\ x_k^1, x_k^2, V_k^1, V_k^2)}{i\phi}\right] d\phi$$

3.4 State-Space Representations

The unobserved volatilities V_k, $\{V_k^i\}_{i,k}$ at each timestep k needs first to be estimated so as to obtain the volatility smile of the Heston models mentioned above. Namundjebo (2016) uses a filtering approach to obtain an estimate for the volatilities in the case of the Standard Heston and Double Heston model. In this study, we extend this approach to the case of the Double Heston stochastic volatility model written on two underlying assets.

V_k is the state variable for the Heston model which is unobserved, similarly, V_k^1 and V_k^2 are the state variables for the Double Heston model written on both one and two underlying assets which are also unobserved. The option prices are taken to be the model observations and the variance processes are taken to be the transition equations.

Therefore, for us to estimate the unobservable factors and the model's parameters, we simply work with the relationship between the stock returns or option prices and the underlying state variables. This is the relationship between the evolution of the measurement equations and the state transition equations. A system of the measurement and transition equations is called the state-space representation of the model.

The measurement noise and the state noise are correlated in the heston models looked at in this study. Cholesky decomposition is used to decorrelate the sources of randomness so as to ensure that for the filters, the process noise and measurement noise are uncorrelated. In order to formulate our models in the state-space representation, we need to specify the state transition equations and the measurement equations. We begin by presenting the state-space form for the Heston model, then we proceed to the Double Heston model and lastly, we extend this to the Double Heston model with two underlying assets.

In the standard Heston model, let the spot price S_k be the observation and the variance V_k be the state. The measurement equation is then represented by the stock price equation and the state transition equation by the variance process. The standard Heston model can be given as

$$\ln\ S_k = \ln\ S_{k-1} + \left(r - q - \frac{1}{2} V_{k-1}\right)\Delta k + \sqrt{V_{k-1}}\sqrt{\Delta k} W_{k-1}$$

$$V_k = V_{k-1} + \kappa(\theta - V_{k-1})\Delta k + \sigma\sqrt{V_{k-1}}\sqrt{\Delta k} Z_{k-1}$$

This system of equations is obtained by applying the Euler method to the Stochastic Differential Equation:

$$dx_k = \mu(k, x_k)dk + \sigma(k, x_k)dW_k$$

dW_k is the Brownian motion, μ and σ are functions depending on S_k and k, over an interval [0, T], and we want to discretize it as $0 = k_1 < k_2 < \cdots < k_m = T$ with increments equally spaced Δt.

Using the Euler discretization we proceed as follows:

$$x_{k-1} - x_{k-1+\Delta k} = \int_{k-1}^{k-1+\Delta k} \mu(r, x_r)dr + \int_{k-1}^{k-1+\Delta k} \sigma(r, x_r)dW_r$$

$$\int_{k-1}^{k-1+\Delta k} \mu(r, x_r)dr \approx \mu(k, x_{k-1}) \int_{k-1}^{k-1+\Delta k} dr$$

$$= \mu(k, x_{k-1})\Delta k$$

$$\int_{k-1}^{k-1+\Delta k} \sigma(r, x_r)dW_r \approx \sigma(k-1, x_{k-1}) \int_{k-1}^{k-1+\Delta k} dW_r$$

$$= \sigma(k-1, x_{k-1})(W_{k-1+\Delta k} - W_{k-1})$$

$$= \sigma(k-1, x_{k-1})\sqrt{\Delta t}W_{k-1}$$

The Euler discretization is given as:

$$x_{k-1} - x_{k-1+\Delta k} = \mu(k, x_{k-1})\Delta k + \sigma(k-1, x_{k-1})\sqrt{\Delta k}W_{k-1}$$

$$\mu(k, x_{k-1}) = r - q - \frac{1}{2}V_{k-1}$$

$$\sigma(k-1, x_{k-1}) = \sqrt{V_{k-1}}$$

Thus

$$\ln\ S_k = \ln\ S_{k-1} + \left(r - q - \frac{1}{2}V_{k-1}\right)\Delta k + \sqrt{V_{k-1}}\sqrt{\Delta k}W_{k-1}$$

Similarly;

$$V_k = V_{k-1} + \kappa(\theta - V_{k-1})\Delta k + \sigma\sqrt{V_{k-1}}\sqrt{\Delta k}Z_{k-1}$$

where W_k and Z_k are correlated. Javaheri (2011) gives an approach to eliminate the correlation between these equations by subtracting from the variance process $f(x_{k-1}, w_k)$ a multiple of the quantity $h(x_k, v_k) - y_k$ which is equal to zero as follows:

$$V_k = V_{k-1} + \kappa(\theta - V_{k-1})\Delta k + \sigma\sqrt{V_{k-1}}\sqrt{\Delta k}Z_{k-1}$$
$$- \rho\sigma\left[\ln S_{k-1} + \left(r - q - \frac{1}{2}V_{k-1}\right)\Delta k + \sqrt{V_{k-1}}\sqrt{\Delta k}W_{k-1} - \ln S_k\right]$$

which gives

$$V_k = V_{k-1} + \left[(\kappa\theta - \rho\sigma(r-q)) - \left(\kappa - \frac{1}{2}\rho\sigma\right)V_{k-1}\right]\Delta k$$
$$+ \rho\sigma\ln\left(\frac{S_k}{S_{k-1}}\right) + \sigma\sqrt{1-\rho^2}\sqrt{V_{k-1}}\sqrt{\Delta k}B_{k-1}.$$

(3.4.1)

where

$$B_k = \frac{1}{\sqrt{1-\rho^2}}(Z_k - \rho W_k)$$

and the measurement equation is

$$y_k = \ln S_k = \ln S_{k-1} + \left(r - q - \frac{1}{2}V_{k-1}\right)\Delta k + \sqrt{V_{k-1}}\sqrt{\Delta k}W_{k-1}.$$

(3.4.2)

B_k and W_k are thus uncorrelated. The state transition equation is represented by Equation 3.4.1.

Li (2013) shows that if we take the spot prices S_k and option prices $C(S_k, K)$ as the observations and the variance V_k as the state, then the measurement equations are represented by

$$\ln S_k = \ln S_{k-1} + \left(r - q - \frac{\rho}{\sigma}\kappa\theta\right)\Delta k + \frac{\rho}{\sigma}V_k + \left[\frac{\rho}{\sigma}(\kappa\Delta k - 1) - \frac{1}{2}\Delta k\right]V_{k-1} + \sqrt{1-\rho^2}\sqrt{\Delta k}\sqrt{V_{k-1}}W_k \tag{3.4.3}$$

$$y_k^0 = g(S_k, V_k, \Theta) + \epsilon_t^0 \tag{3.4.4}$$

where y_k^0 is the observable option prices, with identical independent distributed measurement errors $\epsilon_t^0 \rightarrow N(0, \sigma_0^2)$, independent of W_k and Z_k, and $g(.)$ is the theoretical option price computed from the Heston model.

The state transition equations are given by the variance processes

$$\begin{pmatrix} V_k \\ V_{k-\Delta k} \end{pmatrix} = \begin{pmatrix} \kappa\theta\Delta k \\ 0 \end{pmatrix} + \begin{pmatrix} 1-\kappa\Delta k & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} V_{k-\Delta k} \\ V_{k-2\Delta k} \end{pmatrix} + \begin{pmatrix} \sigma\sqrt{\Delta k V_{k-\Delta k}} \\ 0 \end{pmatrix} Z_k$$

For the Double Heston model, the system equations are

$$\ln S_k = \ln S_{k-1} + \left(r - q - \frac{1}{2}(V_{k-1}^1 + V_{k-1}^2)\right)\Delta k + \sqrt{V_{k-1}^1}\sqrt{\Delta k}W_{k-1}^1 + \sqrt{V_{k-1}^2}\sqrt{\Delta k}W_{k-1}^2,$$

$$V_k^1 = V_{k-1}^1 + \kappa_1(\theta_1 - V_{k-1}^1)\Delta k + \sigma_1\sqrt{V_{k-1}^1}\sqrt{\Delta k}Z_{k-1}^1,$$

$$V_k^2 = V_{k-1}^2 + \kappa_2(\theta_2 - V_{k-1}^2)\Delta k + \sigma_2\sqrt{V_{k-1}^2}\sqrt{\Delta k}Z_k^2. \tag{3.4.5}$$

Taking the observation to be the spot price S_k and the states to be the variance processes V_k^1, V_k^2, then the measurement equation is represented by the stock price $\ln S_k$ in Equation 3.4.5 and the transition equations by the variance processes V_k^1, V_k^2 in Equation 3.3.5. The problem we face when using these equations, the process noise and the measurement noise are correlated, $d[W^1, Z^1]_k = \rho_1 dk$ and $d[W^2, Z^2]_k = \rho_2 dk$. However, for the filtering the process and the measurement noises must be uncorrelated.

By Ito's Lemma, we let $x_k = ln\left(\frac{S_k}{S_{k-1}}\right)$. This implies that

$$dx_k = \left((r-q) - \frac{1}{2}(V_k^1 + V_k^2)\right)\mathrm{d}k + \sqrt{V_k^1}dW_k^1 + \sqrt{V_k^2}dW_k^2 \tag{3.4.6}$$

Making use of Cholesky decomposition, let

$$dW_k^1 = \rho_1 dZ_k^1 + \sqrt{1-\rho_1^2}\, d\tilde{Z}_k^1$$

$$dW_k^2 = \rho_2 dZ_k^2 + \sqrt{1-\rho_2^2}\, d\tilde{Z}_k^2$$

Where $d[Z^1, \tilde{Z}^1] = d[Z^2, \tilde{Z}^2] = 0$

Substituting dW_k^1, dW_k^2 in Equation 3.4.6, we get

$$dx_k = \left((r-q) - \frac{1}{2}(V_k^1 + V_k^2)\right)\mathrm{d}k + \sqrt{V_k^1}\left(\rho_1 dZ_k^1 + \sqrt{1-\rho_1^2}\, d\tilde{Z}_k^1\right) + \sqrt{V_k^2}\left(\rho_2 dZ_k^2 + \sqrt{1-\rho_2^2}\, d\tilde{Z}_k^2\right)$$
$$= \left((r-q) - \frac{1}{2}(V_k^1 + V_k^2)\right)\mathrm{d}k + \rho_1\sqrt{V_k^1}dZ_k^1 + \sqrt{1-\rho_1^2}\sqrt{V_k^1}d\tilde{Z}_k^1 +$$
$$+ \rho_2\sqrt{V_k^2}dZ_k^2 + \sqrt{1-\rho_2^2}\sqrt{V_k^2}d\tilde{Z}_k^2. \tag{3.4.7}$$

From Equation 3.2.1, we know that

$$\sqrt{V_k^1}dZ_k^1 = \frac{1}{\sigma_1}(dV_k^1 - \kappa_1(\theta_1 - V_k^1)dk) \tag{3.4.8}$$

$$\sqrt{V_k^2}dZ_k^2 = \frac{1}{\sigma_2}(dV_k^2 - \kappa_2(\theta_2 - V_k^2)dk) \tag{3.4.9}$$

By substituting Equation 3.4.8 and 3.4.9 into 3.4.7, we get

$$dx_k = \left(r - q - \frac{1}{2}(V_k^1 + V_k^2)\right)\mathrm{d}k + \frac{\rho_1}{\sigma_1}(dV_k^1 - \kappa_1(\theta_1 - V_k^1)dk) + \sqrt{1-\rho_1^2}\sqrt{V_k^1}d\tilde{Z}_k^1$$
$$+ \frac{\rho_2}{\sigma_2}(dV_k^2 - \kappa_2(\theta_2 - V_k^2)dk) + \sqrt{1-\rho_2^2}\sqrt{V_k^2}d\tilde{Z}_k^2.$$

Using a similar approach to Li (2013), we obtain:

$$\ln S_k = \ln S_{k-1} + \left(r - q - \frac{\rho_1}{\sigma_1}\kappa_1\theta_1 - \frac{\rho_2}{\sigma_2}\kappa_2\theta_2\right)\Delta k + \frac{\rho_1}{\sigma_1}V_k^1 + + \frac{\rho_2}{\sigma_2}V_k^2 + \left(\frac{\rho_1}{\sigma_1}(\kappa_1\Delta k - 1) - \frac{1}{2}\Delta k\right)V_{k-1}^1$$
$$+ \left(\frac{\rho_2}{\sigma_2}(\kappa_2\Delta k - 1) - \frac{1}{2}\Delta k\right)V_{k-1}^2 + \sqrt{1-\rho_1^2}\sqrt{V_{k-1}^1}\sqrt{\Delta k}\,\tilde{Z}_k^1 + \sqrt{1-\rho_2^2}\sqrt{V_{k-1}^2}\sqrt{\Delta k}\,\tilde{Z}_k^2. \tag{3.4.10}$$

which is the measurement equation.

The state transition equations are:

$$\begin{pmatrix} V_k^1 \\ V_k^2 \end{pmatrix} = \begin{pmatrix} V_{k-1}^1 + \kappa_1(\theta_1 - V_{k-1}^1)\Delta k + \sigma_1\sqrt{V_{k-1}^1}\sqrt{\Delta k}\, Z_{k-1}^1 \\ V_{k-1}^2 + \kappa_2(\theta_2 - V_{k-1}^2)\Delta k + \sigma_2\sqrt{V_{k-1}^2}\sqrt{\Delta k}\, Z_{k-1}^2 \end{pmatrix} \tag{3.4.11}$$

Clearly, the measurement noise $\tilde{Z}_k^1$ and $\tilde{Z}_k^2$ from in Equation 3.4.10 are uncorrelated to the states noise Z_k^1, Z_k^2 in Equation 3.4.11.

A system is said to be observable if it allows its states estimation from the measurement equation(s). A check therefore needs to be done to see whether the measurement equation given in Equation 3.4.10 and the states in Equation 3.4.11 form an observable system.

Under the Double Heston model written on two underlying assets, the system equations are

$$\ln S_k^1 = \ln S_{k-1}^1 + \left(r_1 - q_1 - \frac{1}{2}(V_{k-1}^1)\right)\Delta k + \sqrt{V_{k-1}^1}\sqrt{\Delta k}W_{k-1}^1,$$

$$\ln S_k^2 = \ln S_{k-1}^2 + \left(r_2 - q_2 - \frac{1}{2}(V_{k-1}^2)\right)\Delta k + \sqrt{V_{k-1}^2}\sqrt{\Delta k}W_{k-1}^2,$$

$$V_k^1 = V_{k-1}^1 + \kappa_1(\theta_1 - V_{k-1}^1)\Delta k + \sigma_1\sqrt{V_{k-1}^1}\sqrt{\Delta k}Z_{k-1}^1,$$

$$V_k^2 = V_{k-1}^2 + \kappa_2(\theta_2 - V_{k-1}^2)\Delta k + \sigma_2\sqrt{V_{k-1}^2}\sqrt{\Delta k}Z_k^2.$$

(3.4.12)

Taking the observation to be the spot prices S_k^1, S_k^2 and the states to be the variance processes V_k^1, V_k^2, then the measurement equation is represented by the stock price $\ln S_k^1, \ln S_k^2$ in Equation 3.4.12 and the transition equations by the variance processes V_k^1, V_k^2 in Equation 3.4.12. The problem we face when using these equations, the process noise and the measurement noise are correlated, $d[W^1, Z^1]_k = \rho_1 dk$ and $d[W^2, Z^2]_k = \rho_2 dk$. However, for the filtering the process and the measurement noises must be uncorrelated.

Making use of Ito's Lemma, let $x_k^i = ln\left(\frac{S_k^i}{S_{k-1}^i}\right)$. This gives us:

$$dx_k^1 = \left(r_1 - q_1 - \frac{1}{2}(V_k^1)\right)dk + \sqrt{V_k^1}dW_k^1,$$

$$dx_k^2 = \left(r_2 - q_2 - \frac{1}{2}(V_k^2)\right)dk + \sqrt{V_k^2}dW_k^2,$$

(3.4.13)

Making use of Cholesky decomposition, let

$$dW_k^1 = \rho_1 dZ_k^1 + \sqrt{1-\rho_1^2}\, d\tilde{Z}_k^1$$

$$dW_k^2 = \rho_2 dZ_k^2 + \sqrt{1-\rho_2^2}\, d\tilde{Z}_k^2$$

Where $d[Z^1, \tilde{Z}^1] = d[Z^2, \tilde{Z}^2] = 0$

Substituting dW_k^1, dW_k^2 in Equation 3.4.13, we get

$$dx_k^1 = \left(r_1 - q_1 - \frac{1}{2}(V_k^1)\right)dk + \sqrt{V_k^1}\left(\rho_1 dZ_k^1 + \sqrt{1-\rho_1^2}\, d\tilde{Z}_k^1\right),$$

$$= \left(r_1 - q_1 - \frac{1}{2}(V_k^1)\right)dk + \rho_1\sqrt{V_k^1}dZ_k^1 + \sqrt{1-\rho_1^2}\sqrt{V_k^1}d\tilde{Z}_k^1,$$

$$dx_k^2 = \left(r_2 - q_2 - \frac{1}{2}(V_k^2)\right)dk + \sqrt{V_k^2}\left(\rho_2 dZ_k^2 + \sqrt{1-\rho_2^2}\, d\tilde{Z}_k^2\right),$$

$$= \left(r_2 - q_2 - \frac{1}{2}(V_k^2)\right)dk + \rho_2\sqrt{V_k^2}dZ_k^2 + \sqrt{1-\rho_2^2}\sqrt{V_k^2}d\tilde{Z}_k^2,$$

(3.4.14)

From Equation 3.2.1, we know that

$$\sqrt{V_k^1}dZ_k^1 = \frac{1}{\sigma_1}(dV_k^1 - \kappa_1(\theta_1 - V_k^1)dk) \quad (3.4.15)$$

$$\sqrt{V_k^2}dZ_k^2 = \frac{1}{\sigma_2}(dV_k^2 - \kappa_2(\theta_2 - V_k^2)dk) \quad (3.4.16)$$

By substituting Equation 3.4.15 and 3.4.16 into 3.4.14, we get

$$dx_k^1 = \left(r_1 - q_1 - \frac{1}{2}(V_k^1)\right)dk + \frac{\rho_1}{\sigma_1}(dV_k^1 - \kappa_1(\theta_1 - V_k^1)dk) + \sqrt{1-\rho_1^2}\sqrt{V_k^1}d\tilde{Z}_k^1,$$

$$dx_k^2 = \left(r_2 - q_2 - \frac{1}{2}(V_k^2)\right)dk + \frac{\rho_2}{\sigma_2}(dV_k^2 - \kappa_2(\theta_2 - V_k^2)dk) + \sqrt{1-\rho_2^2}\sqrt{V_k^2}d\tilde{Z}_k^2,$$

Using a similar approach to Li (2013), we obtain:

$$\ln S_k^1 = \ln S_{k-1}^1 + \left(r_1 - q_1 - \frac{\rho_1}{\sigma_1}\kappa_1\theta_1\right)\Delta k + \frac{\rho_1}{\sigma_1}V_k^1 + \left(\frac{\rho_1}{\sigma_1}(\kappa_1\Delta k - 1) - \frac{1}{2}\Delta k\right)V_{k-1}^1 + \sqrt{1-\rho_1^2}\sqrt{V_{k-1}^1}\sqrt{\Delta k}\,\tilde{Z}_k^1,$$

$$\ln S_k^2 = \ln S_{k-1}^2 + \left(r_2 - q_2 - \frac{\rho_2}{\sigma_2}\kappa_2\theta_2\right)\Delta k + \frac{\rho_2}{\sigma_2}V_k^2 + \left(\frac{\rho_2}{\sigma_2}(\kappa_2\Delta k - 1) - \frac{1}{2}\Delta k\right)V_{k-1}^2 + \sqrt{1-\rho_2^2}\sqrt{V_{k-1}^2}\sqrt{\Delta k}\,\tilde{Z}_k^2. \quad (3.4.17)$$

Equation 3.4.17 are the measurement equations.

The state transition equations are:

$$\begin{pmatrix} V_k^1 \\ V_k^2 \end{pmatrix} = \begin{pmatrix} V_{k-1}^1 + \kappa_1(\theta_1 - V_{k-1}^1)\Delta k + \sigma_1\sqrt{V_{k-1}^1}\sqrt{\Delta k}\,Z_{k-1}^1 \\ V_{k-1}^2 + \kappa_2(\theta_2 - V_{k-1}^2)\Delta k + \sigma_2\sqrt{V_{k-1}^2}\sqrt{\Delta k}\,Z_{k-1}^2 \end{pmatrix} \quad (3.3.18)$$

Clearly, the measurement noise $\tilde{Z}_k^1$ and $\tilde{Z}_k^2$ from in Equation 3.3.17 are uncorrelated to the states noise Z_k^1, Z_k^2 in Equation 3.3.18.

A system is said to be observable if it allows its states estimation from the measurement equation(s). A check therefore needs to be done to see whether the measurement equation given in Equation 3.3.17 and the states in Equation 3.3.18 form an observable system.

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