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On Derivations of Black-Scholes Greek Letters

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Abstract
Each Greek letter of an option measures the sensitivity of an option price with respect to the change in the value of a given underlying parameter such as underlying asset's price. This article provides simple derivations of often-used five Greek letters for European call and put options within the Balck-scholes model framework. Each proof of these Greek letters bypasses complicated mathematical calculations, it is relatively simple and easy to follow. Furthermore, some calculation examples for Greek letters have been given.
Keywords: Black-Scholes option pricing model, Call option, Put option, Greek letters

1. Introduction

Often-mentioned Greek letters of Delta, Theta, Gamma, Vega and Rho in option pricing are generally defined as the sensitivities of an option price relative to changes in the value of either a state variable or a parameter (Hull, 2009). Each of them measures a different dimension to the risk in an option position and, by analysing Greek letters, financial institutions who sell option products to their clients can effectively manage their risk which a practitioner often encounters since the options usually do not correspond to the standardized products traded by exchanges. The well-known Greek letter of Delta (i.e. the option hedge ratio) that is defined as the rate of change of option price to the underlying price, for example, is widely used to determine the number of option contracts for achieving a neutral position in a portfolio.

The celebrated Black-Scholes (Black and Scholes, 1973) model offers an elegant and effective way for option pricing and option hedging since it can give an analytic solution for option price, as well as Greek letters, even though this model could make certain pricing bias in realistic market. The black-Scholes formula thus has been regarded as a benchmark for option valuation and option hedging, and accepted by many financial professionals including practitioners who seek to manage their risk exposure. Typically, an option trader would use the Greek letters under Black-Scholes framework (Black-Scholes Greeks) as a benchmark for properly adjusting option position so that all risks are acceptable.

Generally the derivations of Black-Scholes Greek letters are quite mathematically involved because the calculations of partial derivatives even complicated integrals are required (Chen et al., 2010). For example, the hedge ratio of Black-Scholes option's Delta is commonly derived either by taking the partial derivative of the option price formula with respect to underlying price via the Chain Law, or instead by differentiating the original formula which expresses the option's value as a discounted risk-neutral expectation. The former needs to calculate all involved complicated partial derivative including the derivative of standard normal distribution function, and the latter involves derivative of an integral due to the discounted risk-neutral expectation, both are not easy to follow. This article provides simple derivations for five Greek letters of call and put options under the Black-Scholes model framework. The proofs are succinct and easily-understood.

The remainder of this article is organized as follows. The Black-Scholes model and option-pricing formulas are provided and then definitions of Greek letters are also given in the following section. Section 3 presents the derivations of Greek letters for Black-Scholes call and put options with some calculation examples. Conclusions are in the final section.

2. Black-Scholes Option Pricing Model and Greek Letters

2.1 Option Pricing Model

S_t For simplicity, and yet without any loss of generality, this article just considers that case in which the underlying asset, say stock, pays no dividends. Assume the price of underlying stock follows a geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t d\omega_t \quad (1)$$

where each of the parameters has its standard meaning. St and ω_t denote the stock's price at time t and standard Wiener process, respectively; μ and σ are respectively the expected growth rate and the standard deviation of returns of underlying stock, both are constants.

Given this stochastic process, the Black-Scholes option pricing formulas can be written as:

$$P = -S_0 N(-d_1) - Xe^{-r\tau} N(d_2), \qquad C = S_0 N(d_1) - Xe^{-r\tau} N(d_2) \quad (2)$$

With

$$d_1 = \frac{\ln(S_0 / X) + (r + \sigma^2 / 2)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 = d_1 - \sigma\sqrt{\tau} \quad (3)$$

where *C* and *P* are the call and put option prices, respectively; S_0, *X*, *r* and τ respectively stand for the current price of underlying stock, option's strike price, annual continuously compounded risk-free interest rate and the time to maturity. $N(\cdot)$ denotes the cumulative distribution function of standard normal distribution with density function $n(d) = N'(d) = \frac{\partial N(d)}{\partial d} = \frac{1}{\sqrt{2\pi}} e^{-\frac{d^2}{2}}$ by definition.

2.2 Greek letters

This subsection gives the definitions of five Greek letters as follows.

Delta: $\Delta = \frac{\partial V}{\partial S_0}$

it measures the rate of change of option value with respect to changes in the underlying asset's price.

Theta: $\Theta = \frac{\partial V}{\partial t} = -\frac{\partial V}{\partial \tau}$

it measures the sensitivity of the option value to the passage of time.

Gamma: $\Gamma = \frac{\partial \Delta}{\partial S_0}$

it measures the rate of change in the delta with respect to changes in the underlying price.

Vega: $\nu = \frac{\partial V}{\partial \sigma}$

it is the rate of change of the option price with respect to the volatility of the underlying stock.

Rho: $\rho = \frac{\partial V}{\partial r}$

it is the rate of change of the option price with respect to the interest rate.

where the parameter *V* denotes the option's value, either *C* for call option or *P* for put option.

3. Derivations of Greek letters

In this section, all the proofs of Greek letters for both call and put options are provided in order. Finally the relationship between Delta, Theta, and Gamma is shown. Also some calculation examples are provided in this section.

3.1 Preliminaries

To derive these Greek letters, the following lemmas are necessary and sufficient. Note that all the notations below are specified in previous section and would not be described again.

Lemma 1 *From the relationship of d_1 and d_2 shown in Equation (3), it holds that:*

$$\frac{\partial d_2}{\partial S_0} = \frac{\partial d_1}{\partial S_0} \quad (4)$$

$$\frac{\partial d_2}{\partial \tau} = \frac{\partial d_1}{\partial \tau} - \frac{1}{2}\frac{\sigma}{\sqrt{\tau}} \quad (5)$$

$$\frac{\partial d_2}{\partial \sigma} = \frac{\partial d_1}{\partial \sigma} - \sqrt{\tau} \quad (6)$$

$$\frac{\partial d_2}{\partial r} = \frac{\partial d_1}{\partial r} \quad (7)$$

Proof: From the given relationship $d_2 = d_1 - \sigma\sqrt{\tau}$ in Equation (3). These Equations (4)-(7) are immediate.

Lemma 2 *The relationship between the values of density function* $n(d)$ *d₂ and d₁ can be expressed as:*

$$S_0 n(d_1) = Xe^{-r\tau} n(d_2) \qquad (8)$$

Proof: First, consider the calculation of (d_2- d_1):

$$\begin{aligned} d_2^2 - d_1^2 &= (d_2 - d_1)(d_2 + d_1) \\ &= (-\sigma\sqrt{\tau})(2d_1 - \sigma\sqrt{\tau}) \\ &= (-\sigma\sqrt{\tau})[\frac{2\ln(S_0/X) + 2(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}} - \sigma\sqrt{\tau}] \\ &= -2[\ln(S_0/X) + r\tau] \end{aligned}$$

The above result now is employed to derive Equation (8). By the definition of density function $n(d) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d^2}{2}}$

$$\begin{aligned} \ln\left(\frac{n(d_1)}{n(d_2)}\right) &= -\frac{d_1^2}{2} + \frac{d_2^2}{2} \\ &= \frac{1}{2}(d_2^2 - d_1^2) \\ &= -[\ln(S_0/X) + r\tau] \quad \text{(by the above result of } d_2\text{- } d_1\text{)} \end{aligned}$$

Taking the exponential form from both sides and rearranging the terms can easily achieve Equation (8). This lemma is proved.

3.2 Proofs of Greek Letters

With two proven lemmas above, the expressions of Greek letters can be derived in order as follows.

Proposition 1 *The expressions of Greek letters for Black-Scholes call and put options are as follows:*

For call option,

$$\Delta = \frac{\partial C}{\partial S_0} = N(d_1) \qquad (9)$$

$$\Theta = -\frac{\partial C}{\partial \tau} = -rXe^{-r\tau} N(d_2) - \frac{\sigma}{2\sqrt{\tau}} S_0 n(d_1) \qquad (10)$$

$$\Gamma = \frac{\partial \Delta}{\partial S_0} = \frac{1}{S_0 \sigma\sqrt{\tau}} n(d_1) \qquad (11)$$

$$\nu = \frac{\partial C}{\partial \sigma} = \sqrt{\tau} S_0 n(d_1) \qquad (12)$$

$$\rho = \frac{\partial C}{\partial r} = \tau Xe^{-r\tau} N(d_2) \qquad (13)$$

For put option, applying the parity relation $P = Xe^{-r\tau} + C - S_0$ *obtains its Greeks,*

$$\Delta = \frac{\partial P}{\partial S_0} = N(d_1) - 1 \qquad (14)$$

$$\Theta = -\frac{\partial P}{\partial \tau} = -\frac{\sigma}{2\sqrt{\tau}} S_0 n(d_1) + rXe^{-r\tau} N(-d_2) \qquad (15)$$

$$\Gamma = \frac{\partial \Delta}{\partial S_0} = \frac{1}{S_0 \sigma\sqrt{\tau}} n(d_1) \qquad (16)$$

$$\nu = \frac{\partial P}{\partial \sigma} = \sqrt{\tau} S_0 n(d_1) \qquad (17)$$

$$\rho = \frac{\partial P}{\partial r} = -\tau Xe^{-r\tau} N(-d_2) \qquad (18)$$

Proof: For call option, recall its price formula $C = S_0 N(d_1) - Xe^{-r\tau} N(d_2)$ given in Formula (2), we have,

i) Derivation of Equation (9):

ii) Derivation of Equation (10):

$$\Theta = -\frac{\partial C}{\partial \tau} = -\frac{\partial [S_0 N(d_1) - Xe^{-r\tau} N(d_2)]}{\partial \tau}$$

$$= -S_0 n(d_1)\frac{\partial d_1}{\partial \tau} + Xe^{-r\tau}[-rN(d_2) + n(d_2)\frac{\partial d_2}{\partial \tau}]$$

$$\overset{by(5)}{=} -S_0 n(d_1)\frac{\partial d_1}{\partial \tau} + Xe^{-r\tau}[-rN(d_2) + n(d_2)(\frac{\partial d_1}{\partial \tau} - \frac{1}{2}\frac{\sigma}{\sqrt{\tau}})]$$

$$= [-S_0 n(d_1) + Xe^{-r\tau} n(d_2)]\frac{\partial d_1}{\partial \tau} + Xe^{-r\tau}[-rN(d_2) - \frac{1}{2}\frac{\sigma}{\sqrt{\tau}} n(d_2)]$$

$$\overset{by(8)}{=} -rXe^{-r\tau} N(d_2) - \frac{1}{2}\frac{\sigma}{\sqrt{\tau}} S_0 n(d_1)$$

iii) Derivation of Equation (11):

$$\Gamma = \frac{\partial \Delta}{\partial S_0} = \frac{\partial N(d_1)}{\partial S_0}$$

$$= n(d_1)\frac{\partial d_1}{\partial S_0}$$

$$= n(d_1)\frac{\partial [\frac{\ln(S_0 / X) + (r + \sigma^2 / 2)\tau}{\sigma\sqrt{\tau}}]}{\partial S_0}$$

$$= \frac{1}{S_0 \sigma \sqrt{\tau}} n(d_1)$$

iv) Derivation of Equation (12):

$$\nu = \frac{\partial C}{\partial \sigma} = \frac{\partial [S_0 N(d_1) - Xe^{-r\tau} N(d_2)]}{\partial \sigma}$$

$$= S_0 n(d_1)\frac{\partial d_1}{\partial \sigma} - Xe^{-r\tau} n(d_2)\frac{\partial d_2}{\partial \sigma}$$

$$\overset{by(6)}{=} S_0 n(d_1)\frac{\partial d_1}{\partial \sigma} - Xe^{-r\tau} n(d_2)[\frac{\partial d_1}{\partial \sigma} - \sqrt{\tau}]$$

$$= [S_0 n(d_1) - Xe^{-r\tau} n(d_2)]\frac{\partial d_1}{\partial \sigma} + \sqrt{\tau} Xe^{-r\tau} n(d_2)$$

$$\overset{by(8)}{=} \sqrt{\tau} S_0 n(d_1)$$

v) Derivation of Equation (13):

$$\rho = \frac{\partial C}{\partial r} = \frac{\partial [S_0 N(d_1) - Xe^{-r\tau} N(d_2)]}{\partial r}$$

$$= S_0 n(d_1)\frac{\partial d_1}{\partial r} - Xe^{-r\tau}[n(d_2)\frac{\partial d_2}{\partial r} - \tau N(d_2)]$$

$$\overset{by(7)}{=} [S_0 n(d_1) - Xe^{-r\tau} n(d_2)]\frac{\partial d_1}{\partial r} + \tau Xe^{-r\tau} N(d_2)$$

$$\overset{by(8)}{=} \tau Xe^{-r\tau} N(d_2)$$

Now turning to the Greek letters for put option.

Note the parity relation for European option $P = Xe^{-r\tau} + C - S_0$ then employ the above results for call option, we can immediately obtain the Greek letters for put option as shown in Equations (14)-(18).

Finally, the relationship between Delta, Theta, and Gamma satisfies the well-known Black-Scholes partial differential equation. It is shown in the following Corollary.

Corollary 1 *With proven Proposition, Delta, Theta, and Gamma satisfies the Black-Scholes partial differential equation,*

$$\frac{\partial V}{\partial t} + rS_0 \frac{\partial V}{\partial S_0} + \frac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 V}{\partial S_0^2} = rV$$

That's,

$$\Theta + rS_0\Delta + \frac{1}{2}\sigma^2 S_0^2 \Gamma = rV \qquad (19)$$

where the parameter V denotes the option's value, either C for call option or P for put option.

Proof: Direct from Proposition and Formula (2).

3.3 Calculation examples of Greek Letters

i) $\Delta = \frac{\partial V}{\partial S_0}$

Geometrically, the delta of one option means the slope of the line at the point which corresponds to current underlying price. Also it can be directly calculated using the equation (9) of $\Delta = N(d_1)$
Suppose a financial institution has sold a European call option expiring within 20 weeks on 1000,000 shares of a non-dividend paying stock, further assume that current stock price is S=$49, strike price is K=$50 with parameters of risk-free interest rate equal to 5% and volatility equal to 20%. By calculating delta ratio, the financial institution can make a delta neutral position to offset changes in stock price. The delta ratio then, by formula $\Delta = N(d_1)$ is 0.522. To get a delta neutral position, the financial institution must hold 0.522 x 100,000 = 52,200 shares of stock. If the stock price increases by $1 at maturity, the option price will increase by $0.522. The financial institution ther efore has a gain of $52,200 (=$1 x 52,200) gain in its stock position, and a $52,200 (=$0.522 x 100,000 shares) loss in its option position. Thus, the financial institution obtains a total payoff of zero. In a similar spirit, when the stock price decreases by $1, the total payoff of the financial institution is also zero.

ii) $\Theta = -\frac{\partial V}{\partial \tau}$

The option value consists of intrinsic value and time value, and the time value decreases as time declines towards maturity. Thus, for a European call, the rate of change of its price with respective to the passage of time, theta, is usually negative. Furthermore, as the time to maturity decreases, the value of call decreases to the intrinsic value. For a put on non-dividend paying stock, however, the theta could be positive. The reason for this is that an in-the-money put can be worth less than its intrinsic value. As time passes, the put value increases to the intrinsic value. From the derived equations (10) and (15), we can also reach the above results.

Since the passage of time on an option is not uncertain, it is unnecessary to make a theta hedge portfolio against the effect of the passage of time.

iii) $\Gamma = \frac{\partial \Delta}{\partial S_0}$

Gamma measures the error from delta hedging caused by the curvature of relationship between underlying asset price and option price so that one can construct a delta and gamma neutral portfolio against price movements between hedge rebalancing. It should be noted that, to maintain the delta neutrality, gamma neutrality usually requires a position in another derivative.

Assume there are two call options on the same underlying asset with parameters of delta and gamma:

$\Delta_1 = 0.50, \Gamma_1 = 0.03; \Delta_2 = 0.40, \Gamma_2 = 0.025$ suppose one hundred of the first options is written (that means n_1= -100) , we can now calculate n_2, the number of second call to buy or sell and n_3, the number of shares for the neutral portfolio by solving the following equations derived from the portfolio value of $\pi_p = n_1c_1 + n_2c_2 + n_3S$,

Delta of portfolio: $\Delta_p = n_1\Delta_1 + n_2\Delta_2 + n_3 = 0$

Gamma of portfolio: $\Gamma_p = n_1\Gamma_1 + n_2\Gamma_2 = 0$, that gives n_2=120, n_3=2.

iv) $\nu = \frac{\partial V}{\partial \sigma}$

As mentioned above, to keep both neutralities, one portfolio generally involves at least two derivatives on the same underlying asset.

For example, a delta-neutral and gamma-neutral portfolio consisting of option I, option II, and underlying asset has a gamma of -3,200 and vega of -2,500. Option I has a delta of 0.3, gamma of 1.2, and vega of 1.5. Option II has a delta of 0.4, gamma of 1.6 and vega of 0.8. Then a new portfolio with both gamma neutrality and vega neutrality can be obtained when choosing proper n_1, the number of option I and n_2, the number of option II according to equations below,

Gamma neutrality: $-3200 + 1.2n_1 + 1.6n_2 = 0$

Vega neutrality: $-2500 + 1.5n_1 + 0.8n_2 = 0$

The solution of these equations is n_1=1000, n_2=1250.

v) $\rho = \frac{\partial V}{\partial r}$

When investing in currency derivative, the investor should consider the factor of rho because the value of derivative on currency could be very sensitive to interest rate.

Assume that a US investment bank has sold a 6-month put option on 100 million Australian Dollar (AUD) with strike price of $1.00, the current spot rate is 1AUD=0.98USD, the interest rate is 6% in the US and the exchange rate volatility is 15% per annum. Then by the equation (18), rho for this put is,

$$\rho = \frac{\partial P}{\partial r} = -\tau X e^{-r\tau} N(-d_2) = -0.5 \times 1 \times e^{-0.06\times 0.5} N(-0.039) = -0.5 \times 0.97 \times 0.484 = -0.235$$

This result indicates that given the domestic interest rate, 1% change in this interest rate will result in an opposite movement of $0.00235 (=0.01 x 0.235) in the value of put option on AUD.

In brief, in this section, Black-Scholes Greek letters for both call and put options are derived in a succinct and easily-understood way. Just as shown from the proof to Proposition, all the derivations only depend on two simple lemmas and do not involve any complicated partial derivative. The proofs are very straightforward owning to Lemma 1 and Lemma 2 and do not even require the derivative or integral of standard normal probability density function which is indispensable in a conventional way. In addition, some application examples have been discussed in this section.

4. Conclusion

In summary, this article presents a simple way to derive five Greek letters for both European call and put options under the Black-Scholes model framework by upfront providing two clear lemmas. The proofs are quite clear and not complicated mathematically, each of them is succinct and easily-understood. Finally the relation between Delta, Theta, and Gamma is checked. Furthermore, some calculation examples of these Greek letters are provided.

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