Research Journal of Finance and Accounting
ISSN 2222-1697 (Paper) ISSN 2222-2847 (Online)
Vol.12, No.21, 2021


# MODELLING STOCK RETURNS USING A NORMAL VARIANCE MEAN MIXTURE WITH FINITE MIXTURE OF WEIGHTED INVERSE GAUSSIAN DISTRIBUTIONS

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**Abstract**

The Normal Weighted Inverse Gaussian distribution arises as a Normal Variance Mean mixture with Weighted Inverse Gaussian mixing distribution. Appropriate choice of the weight for Inverse Gaussian gives rise to finite mixture of special cases of Generalized Inverse Gaussian (GIG) distributions. This article deals with a kind of weight which lead to a finite mixture between a GIG distribution of indexes $-\frac{1}{2}$ and $-\frac{3}{2}$. Due to the complexity of the likelihood, direct maximization is difficult. We propose an EM type Algorithm to overcome the computational burden occurring when standard numerical techniques are used. The algorithm utilises a subtle approach which is not necessarily based on explicit solution to the normal equations. The iterative scheme is based on a representations of the normal equations. Applications to data sets concerning the $s\&p500$, Range Resources Corporation (RRC) and Shares of Chevron (CVX) are given.

**Keywords:** Finite Mixture, Weighted Distribution, Mixed Model, EM-algorithm.

## 1 Introduction

Mixture models provide a general framework for deriving models applicable in situations when simple models fail (Karlis, 2). Normal Variance-Mean mixtures assume that the variance is not fixed but it is also related to the mean. When the Generalized Inverse Gaussian distribution is the mixing distribution we obtain the Generalized Hyperbolic Distribution (GHD) introduced by Barndorff-Nielsen (1977). The Generalized Inverse Gaussian is a three parameter distribution denoted as $GIG(\lambda, \delta, \gamma)$. $\lambda$ is the index parameter and gives a number of special cases when it takes specified values. Different cases for GIG have been considered in literature (see e.g., Barndorff-Nielsen, 1997; Eberlein and Keller, 1995; Aas and Haff, 2006). One way of extending this work is to consider finite cases of the special cases of GIG. These finite cases can also be expressed as weighted Inverse Gaussian distributions. The concept of weighted distributions was introduced by Fisher (1934) and later developed by Lambert et. al., (1978). Gupta and Kundu (2011) developed a versatile lifetime model as a finite mixture of Inverse Gaussian (IG) and Reciprocal Inverse Gaussian distributions. This article consider a finite mixture of IG and $GIG(-\frac{3}{2}, \delta, \gamma)$ which is shown to be a weighted IG distribution. The mixture is then used as a mixing distribution for Normal Variance-Mean mixture.

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The maximum likelihood parameters estimates of the proposed model are obtained via the Expectation-Maximization (EM)-algorithm introduced by Dempster et al., (1977). The algorithm utilises a subtle approach which is not necessarily based on explicit solution to the normal equations. The iterative scheme is based on a representations of the normal equations. Applications to data sets concerning the s& p500, Range Resources Corporation (RRC) and Shares of Chevron (CVX) are given.

## 2 Proposed Mixing Distribution

**Definition:**Let $Z$ be a random variable with pdf $f(z)$. Then a function $w(Z)$ is also a random variable with expectation

$$E[w(Z)] = \int_{-\infty}^{\infty} w(z)f(z)\,dz$$
$$\therefore 1 = \int_{-\infty}^{\infty} \frac{w(z)}{E[w(Z)]} f(z)\,dz$$

Thus we have

$$f_W(z) \quad = \tfrac{w(z)}{E[w(Z)]} f(z), -\infty < x < \infty \tag{1}$$

Now, suppose $Z \sim IG(\gamma, \delta)$ the Inverse Gaussian distribution with parameters $\gamma$ and $\delta$ and probability density function given by

$$f(z) = \frac{\delta}{\sqrt{2\pi}} \exp(\delta\gamma) z^{-\frac{3}{2}} \exp\left(-\frac{1}{2}\left(\frac{\delta^2}{z} + \gamma^2 z\right)\right) \tag{2}$$

. Consider the weight

$$w(Z) = \left(1 + \frac{Z^{-1}}{1+\delta\gamma}\right) \tag{3}$$

The weighted model becomes

$$g(z) = \frac{\delta^2}{1+\delta^2}\left(1 + \frac{Z^{-1}}{1+\delta\gamma}\right) f(z) \tag{4}$$

Which can also be obtained as a finite mixture of Inverse Gaussian and $GIG(-\frac{3}{2}, \delta, \gamma)$.

i.e,

$$pGIG(-\frac{1}{2}, \delta, \gamma) + (1-p)GIG(-\frac{3}{2}, \delta, \gamma) \tag{5}$$

with

$$p = \frac{\delta^2}{1+\delta^2} \tag{6}$$

 

## 3 The Mixed Model

A stochastic representation of a Normal Variance-Mean mixture is given by letting Let

$$X = \mu + \beta Z + \sqrt{Z} Y$$

$$where$$

$$Y \sim N(0,1)$$

and Z, independent of Y, is a positive random variable.

If F(x) is a cdf of X, then

$$\begin{aligned} F(x) &= prob\{X \le x\} \\ &= \{Y \le \frac{x-\mu-\beta z}{\sqrt{z}}, 0 < z < \infty\} \\ &= \int_0^\infty \int_{-\infty}^{\frac{x-\mu-\beta z}{\sqrt{z}}} \phi(y) g(z)\, dy\, dz \\ &= \int_0^\infty \Phi\Big(\frac{x-\mu-\beta z}{\sqrt{z}}\Big) g(z)\, dz \end{aligned}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are pdf and cdf of a standard normal distribution, respectively.

$$\begin{aligned} \therefore f(x) &= \int_0^\infty \frac{1}{\sqrt{z}} \phi\Big(\frac{x-\mu-\beta z}{\sqrt{z}}\Big) g(z)\, dz \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi z}} e^{-\frac{(x-(\mu+\beta z))^2}{2z}} g(z)\, dz \end{aligned} \tag{7}$$

Thus we have a hierarchical representation as

$$X/Z = z \sim N(\mu + \beta z, z) \tag{8}$$

being the conditional pdf and g(z) the mixing distribution. Suppose the mixing distribution is defined by equation (4) we obtain

$$\begin{aligned} f(x) &= \frac{\delta^3 e^{\delta\gamma}}{2\pi(1+\delta^2)} e^{\beta(x-\mu)} \int_0^\infty (1 + \frac{z^{-1}}{1+\delta\gamma}) z^{-2} e^{-\frac{1}{2}\left(\alpha^2 z + \frac{\delta^2 q(x)}{z}\right)}\, dz \\ &= \frac{\delta^3 e^{\delta\gamma}}{2\pi(1+\delta^2)} e^{\beta(x-\mu)} \int_0^\infty (z^{-1-1} + \frac{z^{-2-1}}{1+\delta\gamma}) e^{-\frac{1}{2}\left(\alpha^2 z + \frac{\delta^2 q(x)}{z}\right)}\, dz \\ therefore \end{aligned} \tag{9}$$

 

$$\begin{aligned}
f(x) &= \frac{\delta^3 e^{\delta\gamma} e^{\beta(x-\mu)}}{\pi(1+\delta^2)} \left\{ \left[\frac{\delta\sqrt{\phi(x)}}{\alpha}\right]^{-1} K_1\big(\alpha\delta\sqrt{\phi(x)}\big) + \left[\frac{\delta\sqrt{\phi(x)}}{\alpha}\right]^{-2} \frac{K_2\big(\alpha\delta\sqrt{\phi(x)}\big)}{1+\delta\gamma} \right\} \\
&= \frac{\delta^3 e^{\delta\gamma} e^{\beta(x-\mu)}}{\pi(1+\delta^2)} \left\{ \frac{\alpha}{\delta\sqrt{\phi(x)}} K_1\big(\alpha\delta\sqrt{\phi(x)}\big) + \frac{\alpha^2}{\delta^2\phi(x)} \frac{K_2\big(\alpha\delta\sqrt{\phi(x)}\big)}{1+\delta\gamma} \right\} \\
&= \frac{\delta e^{\delta\gamma} e^{\beta(x-\mu)}}{\pi(1+\delta^2)\phi(x)} \left\{ \alpha\delta\sqrt{\phi(x)} K_1\big(\alpha\delta\sqrt{\phi(x)}\big) + \frac{\alpha^2}{1+\delta\gamma} K_2\big(\alpha\delta\sqrt{\phi(x)}\big) \right\}
\end{aligned}$$

where

$$\gamma = \sqrt{\alpha^2 - \beta^2}$$

**The log-likelihood function**

$$\begin{aligned}
l &= \sum_{i=1}^{n} \log f(x_i) \\
&= \sum_{i=1}^{n} \left\{ \log\delta + \delta\gamma + \beta(x_i - \mu) - \log(\pi(1+\delta^2)) - \log(\phi(x_i)) + \log\left\{ \alpha\delta\sqrt{\phi(x)} K_1\big(\alpha\delta\sqrt{\phi(x)}\big) + \right.\right. \\
&\quad \left.\left. \frac{\alpha^2}{1+\delta\gamma} K_2\big(\alpha\delta\sqrt{\phi(x)}\big) \right\} \right\} \\
&= n\log\delta + n\delta\gamma + \beta\sum_{i=1}^{n} x_i - n\beta\mu - n\log(\pi(1+\delta^2)) - \sum_{i=1}^{n} \log(\phi(x_i)) + \sum_{i=1}^{n} \log\left\{ \alpha\delta\sqrt{\phi(x)} K_1\big(\alpha\delta\sqrt{\phi(x)}\big) + \right. \\
&\quad \left. \frac{\alpha^2}{1+\delta\gamma} K_2\big(\alpha\delta\sqrt{\phi(x)}\big) \right\} \qquad (10)
\end{aligned}$$

**Posterior Expectation**

$$\begin{aligned}
E(Z/X) &= \frac{\int_0^\infty z f(x/z) g(z)\, dz}{\int_0^\infty f(x/z) g(z)\, dz} \\
&= \frac{\int_0^\infty z(1 + \frac{z^{-1}}{1+\delta\gamma}) z^{-2} e^{-\frac{1}{2}\left(\alpha^2 z + \frac{\delta^2\phi(x)}{z}\right)}\, dz}{\int_0^\infty (1 + \frac{z^{-1}}{1+\delta\gamma}) z^{-2} e^{-\frac{1}{2}\left(\alpha^2 z + \frac{\delta^2\phi(x)}{z}\right)}\, dz}
\end{aligned}$$

$$\begin{aligned}
E(Z/X) &= \frac{\int_0^\infty (z^{0-1} + \frac{z^{-1-1}}{1+\delta\gamma})e^{-\frac{1}{2}\left(\alpha^2 z + \frac{\delta^2\phi(x)}{z}\right)}\,dz}{\int_0^\infty (z^{-1-1} + \frac{z^{-2-1}}{1+\delta\gamma})e^{-\frac{1}{2}\left(\alpha^2 z + \frac{\delta^2\phi(x)}{z}\right)}\,dz} \\
&= \frac{K_0\big(\alpha\delta\sqrt{\phi(x)}\big) + \left(\frac{\delta\sqrt{\phi(x)}}{\alpha}\right)^{-1}\frac{K_{-1}\left(\alpha\delta\sqrt{\phi(x)}\right)}{1+\delta\gamma}}{\left(\frac{\delta\sqrt{\phi(x)}}{\alpha}\right)^{-1}K_{-1}\big(\alpha\delta\sqrt{\phi(x)}\big) + \left(\frac{\delta\sqrt{\phi(x)}}{\alpha}\right)^{-2}\frac{K_{-2}\left(\alpha\delta\sqrt{\phi(x)}\right)}{1+\delta\gamma}} \\
&= \frac{K_0\big(\alpha\delta\sqrt{\phi(x)}\big) + (\frac{\alpha}{\delta\sqrt{\phi(x)}})\frac{K_1\left(\alpha\delta\sqrt{\phi(x)}\right)}{1+\delta\gamma}}{(\frac{\alpha}{\delta\sqrt{\phi(x)}})K_1\big(\alpha\delta\sqrt{\phi(x)}\big) + (\frac{\alpha^2}{\delta^2\phi(x)})\frac{K_2\left(\alpha\delta\sqrt{\phi(x)}\right)}{1+\delta\gamma}} \\
&= \frac{\delta^2\phi(x)K_0\big(\alpha\delta\sqrt{\phi(x)}\big) + (\frac{\alpha\delta\sqrt{\phi(x)}}{1+\delta\gamma})K_1\big(\alpha\delta\sqrt{\phi(x)}\big)}{\alpha\delta\sqrt{\phi(x)}K_1\big(\alpha\delta\sqrt{\phi(x)}\big) + (\frac{\alpha^2}{1+\delta\gamma})K_2\big(\alpha\delta\sqrt{\phi(x)}\big)} \\
&= \frac{(1+\delta\gamma)\delta^2\phi(x)K_0\big(\alpha\delta\sqrt{\phi(x)}\big) + \alpha\delta\sqrt{\phi(x)}K_1\big(\alpha\delta\sqrt{\phi(x)}\big)}{(1+\delta\gamma)\alpha\delta\sqrt{\phi(x)}K_1\big(\alpha\delta\sqrt{\phi(x)}\big) + \alpha^2 K_2\big(\alpha\delta\sqrt{\phi(x)}\big)} \\
&= \frac{\alpha\delta\sqrt{\phi(x)}K_1\big(\alpha\delta\sqrt{\phi(x)}\big) + (1+\delta\gamma)\delta^2\phi(x)K_0\big(\alpha\delta\sqrt{\phi(x)}\big)}{(1+\delta\gamma)\alpha\delta\sqrt{\phi(x)}K_1\big(\alpha\delta\sqrt{\phi(x)}\big) + \alpha^2 K_2\big(\alpha\delta\sqrt{\phi(x)}\big)} \qquad (11)
\end{aligned}$$

Similarly

$$\begin{aligned}
E\Big(\frac{1}{Z}/X\Big) &= \frac{\int_0^\infty z^{-1} f(x/z)g(z)\,dz}{\int_0^\infty f(x/z)g(z)\,dz} \\
&= \frac{\int_0^\infty z^{-1}(1 + \frac{z^{-1}}{1+\delta\gamma})z^{-2}e^{-\frac{1}{2}\left(\alpha^2 z + \frac{\delta^2\phi(x)}{z}\right)}\,dz}{\int_0^\infty (1 + \frac{z^{-1}}{1+\delta\gamma})z^{-2}e^{-\frac{1}{2}\left(\alpha^2 z + \frac{\delta^2\phi(x)}{z}\right)}\,dz} \\
&= \frac{\int_0^\infty (z^{-2-1} + \frac{z^{-3-1}}{1+\delta\gamma})e^{-\frac{1}{2}\left(\alpha^2 z + \frac{\delta^2\phi(x)}{z}\right)}\,dz}{\int_0^\infty (z^{-1-1} + \frac{z^{-2-1}}{1+\delta\gamma})e^{-\frac{1}{2}\left(\alpha^2 z + \frac{\delta^2\phi(x)}{z}\right)}\,dz} \\
&= \frac{\left(\frac{\delta\sqrt{\phi(x)}}{\alpha}\right)^{-2}K_2\big(\alpha\delta\sqrt{\phi(x)}\big) + \left(\frac{\delta\sqrt{\phi(x)}}{\alpha}\right)^{-3}\frac{K_3\left(\alpha\delta\sqrt{\phi(x)}\right)}{1+\delta\gamma}}{\left(\frac{\delta\sqrt{\phi(x)}}{\alpha}\right)^{-1}K_1\big(\alpha\delta\sqrt{\phi(x)}\big) + \left(\frac{\delta\sqrt{\phi(x)}}{\alpha}\right)^{-2}\frac{K_2\left(\alpha\delta\sqrt{\phi(x)}\right)}{1+\delta\gamma}} \\
&= \frac{\frac{\alpha^2}{\delta^2\phi(x)}K_2\big(\alpha\delta\sqrt{\phi(x)}\big) + \frac{\alpha^3}{\delta^3(\sqrt{\phi(x)})^3}\frac{K_3\left(\alpha\delta\sqrt{\phi(x)}\right)}{1+\delta\gamma}}{\frac{\alpha}{\delta\sqrt{\phi(x)}K_1\left(\alpha\delta\sqrt{\phi(x)}\right)} + \frac{\alpha^2}{\delta^2\phi(x)}\frac{K_2\left(\alpha\delta\sqrt{\phi(x)}\right)}{1+\delta\gamma}} \\
&= \frac{\alpha^2\delta\sqrt{\phi(x)}K_2\big(\alpha\delta\sqrt{\phi(x)}\big) + \frac{\alpha^3 K_3\left(\alpha\delta\sqrt{\phi(x)}\right)}{1+\delta\gamma}}{\alpha\delta^2\phi(x)K_1\big(\alpha\delta\sqrt{\phi(x)}\big) + \alpha^2\delta\sqrt{\phi(x)}\frac{K_2\left(\alpha\delta\sqrt{\phi(x)}\right)}{1+\delta\gamma}}
\end{aligned}$$

Therefore

$$
\begin{aligned}
E\left(\frac{1}{Z}/X\right) &= \frac{(1+\delta\gamma)\alpha^2\delta\sqrt{\phi(x)}K_2(\alpha\delta\sqrt{\phi(x)})+\alpha^3K_3(\alpha\delta\sqrt{\phi(x)})}{(1+\delta\gamma)\alpha\delta^2\phi(x)K_1(\alpha\delta\sqrt{\phi(x)})+\alpha^2\delta\sqrt{\phi(x)}K_2(\alpha\delta\sqrt{\phi(x)})} \\
&= \frac{(1+\delta\gamma)\alpha\delta\sqrt{\phi(x)}K_2(\alpha\delta\sqrt{\phi(x)})+\alpha^2K_3(\alpha\delta\sqrt{\phi(x)})}{(1+\delta\gamma)\delta^2\phi(x)K_1(\alpha\delta\sqrt{\phi(x)})+\alpha\delta\sqrt{\phi(x)}K_2(\alpha\delta\sqrt{\phi(x)})} \\
&= \frac{(1+\delta\gamma)\alpha\delta\sqrt{\phi(x)}K_2(\alpha\delta\sqrt{\phi(x)})+\alpha^2K_3(\alpha\delta\sqrt{\phi(x)})}{\alpha\delta\sqrt{\phi(x)}K_2(\alpha\delta\sqrt{\phi(x)})+(1+\delta\gamma)\delta^2\phi(x)K_1(\alpha\delta\sqrt{\phi(x)})} \qquad (12)
\end{aligned}
$$

## 4 Expectation Maximization (EM) algorithm

EM algorithm is a powerful technique for maximum likelihood estimation for data containing missing values or data that can be considered as containing missing values. It was introduced by Dempster et al. (1977).

Assume that the true data are made of an observed part X and unobserved part Z. This then ensures the log likelihood of the complete data $(x_i, z_i)$ for $i = 1, 2, 3, ..., n$ factorizes into two parts (Kostas 2007). i.e.,

$$
\begin{aligned}
\log L &= \log\prod_{i=1}^{n} f(x_i/z_i) + \log\prod_{i=1}^{n} g(z_i) \\
&= \sum_{i=1}^{n}\log f(x_i/z_i) + \sum_{i=1}^{n}\log g(z_i)
\end{aligned}
$$

$where$

$$l_1 = \sum_{i=1}^{n}\log f(x_i/z_i)$$

$and$

$$l_2 = \sum_{i=1}^{n}\log g(z_i)$$

Karlis (2002) applied EM algorithm to mixtures which he considered to consist of two parts; the conditional pdf is for observed data and the mixing distribution is based on an unobserved data, the missing values.

### 4.1 M-Step for the Mixing Distribution

$$
\begin{aligned}
g(z) &= \frac{\delta^2}{1+\delta^2}\left(1+\frac{1}{1+\delta\gamma}\frac{1}{z}\right)g_1(z) \\
&= \frac{\delta^2}{1+\delta^2}\left(1+\frac{1}{1+\delta\gamma}\frac{1}{z}\right)\frac{\delta e^{\delta\gamma}}{\sqrt{2\pi}}z^{-\frac{3}{2}}e^{-\frac{1}{2}\left(\gamma^2 z+\frac{\delta^2}{z}\right)} \\
&= \frac{\delta^3}{1+\delta^2}\frac{e^{\delta\gamma}}{\sqrt{2\pi}}\left(1+\frac{1}{1+\delta\gamma}\frac{1}{z}\right)z^{-\frac{3}{2}}e^{-\frac{1}{2}\left(\gamma^2 z+\frac{\delta^2}{z}\right)} \qquad (13)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
l_2 &= \sum_{i=1}^{n}\log g(z_i) \\
&= \sum_{i=1}^{n}\left\{3\log\delta+\delta\gamma-\log(1+\delta^2)-\frac{1}{2}\log(2\pi)-\frac{3}{2}\log z_i+\log\left(1+\frac{1}{1+\delta\gamma}\frac{1}{z_i}\right)-\frac{\gamma^2}{2}z_i-\frac{\delta^2}{2}\frac{1}{z_i}\right\} \\
&= 3n\log\delta+n\delta\gamma-n\log(1+\delta^2)-\frac{n}{2}\log(2\pi)-\frac{3}{2}\sum_{i=1}^{n}\log z_i+\sum_{i=1}^{n}\log\left(1+\frac{1}{1+\delta\gamma}\frac{1}{z_i}\right) \\
&\quad -\frac{\gamma^2}{2}\sum_{i=1}^{n}z_i-\frac{\delta^2}{2}\sum_{i=1}^{n}\frac{1}{z_i} \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
\therefore \frac{\partial}{\partial\gamma}l_2 &= n\delta+\sum_{i=1}^{n}\frac{-\frac{\delta}{z_i}(1+\delta\gamma)^{-2}}{1+\frac{1}{1+\delta\gamma}\frac{1}{z_i}}-\gamma\sum_{i=1}^{n}z_i \\
&= n\delta-\sum_{i=1}^{n}\frac{1}{(1+\delta\gamma)^2 z_i+(1+\delta\gamma)}-\gamma\sum_{i=1}^{n}z_i \\
&= n\delta-\frac{\delta}{(1+\delta\gamma)}\sum_{i=1}^{n}\frac{1}{1+(1+\delta\gamma)z_i}-\gamma\sum_{i=1}^{n}z_i
\end{aligned}
$$

$$
\begin{aligned}
\therefore \frac{\partial}{\partial\gamma}l_2 &= 0 \Longrightarrow \gamma\sum_{i=1}^{n}z_i = n\delta-\frac{\delta}{(1+\delta\gamma)}\sum_{i=1}^{n}\frac{1}{1+(1+\delta\gamma)z_i} \\
\therefore \hat{\gamma} &= \frac{\delta\left[n-\frac{1}{(1+\delta\gamma)}\sum_{i=1}^{n}\frac{1}{1+(1+\delta\gamma)z_i}\right]}{\sum_{i=1}^{n}z_i} \qquad (15)
\end{aligned}
$$

Next

$$
\begin{aligned}
\frac{\partial}{\partial\delta}l_2 &= \frac{3n}{\delta} + n\gamma - \frac{2n\delta}{1+\delta^2} + \sum_{i=1}^{n} \frac{-\frac{1}{z_i}\gamma(1+\delta\gamma)^{-2}}{1+\frac{1}{1+\delta\gamma}\frac{1}{z_i}} - \delta\sum_{i=1}^{n}\frac{1}{z_i} \\
&= \frac{3n}{\delta} + n\gamma - \frac{2n\delta}{1+\delta^2} - \frac{\gamma}{1+\delta\gamma}\sum_{i=1}^{n}\frac{1}{1+(1+\delta\gamma)z_i} - \delta\sum_{i=1}^{n}\frac{1}{z_i} \\
&= \frac{3n}{\delta} - \frac{2n\delta}{1+\delta^2} + \gamma\left[n - \frac{1}{1+\delta\gamma}\sum_{i=1}^{n}\frac{1}{1+(1+\delta\gamma)z_i}\right] - \delta\sum_{i=1}^{n}\frac{1}{z_i} \\
&= \frac{3n+n\delta^2}{\delta(1+\delta^2)} + \gamma\left[n - \frac{1}{1+\delta\gamma}\sum_{i=1}^{n}\frac{1}{1+(1+\delta\gamma)z_i}\right] - \delta\sum_{i=1}^{n}\frac{1}{z_i} \\
\therefore \frac{\partial}{\partial\delta}l_2 = 0 \Longrightarrow \delta &= \frac{\frac{3n+n\delta^2}{\delta(1+\delta^2)} + \gamma\left[n - \frac{1}{1+\delta\gamma}\sum_{i=1}^{n}\frac{1}{1+(1+\delta\gamma)z_i}\right]}{\sum_{i=1}^{n}\frac{1}{z_i}} \qquad (16)
\end{aligned}
$$

## E-Step

Values of random variables $Z_i$ and $\frac{1}{Z_i}$ are not known. So we estimate them by considering posterior expectations

$$E(Z_i/X_i) \text{ and } E(\tfrac{1}{Z_i}/X_i)$$

Let

$$s_i = E(Z_i/X_i) \text{ and } w_i = E(\tfrac{1}{Z_i}/X_i)$$

## Iterations

$$s_i^{(k)} = \frac{\alpha^{(k)}\delta^{(k)}\sqrt{\phi^{(k)}(x_i)}K_1\big(\alpha^{(k)}\delta^{(k)}\sqrt{\phi^{(k)}(x_i)}\big) + (1+\delta^{(k)}\gamma^{(k)})\delta^2\phi^{(k)}(x_i)K_0\big(\alpha^{(k)}\delta^{(k)}\sqrt{\phi^{(k)}(x_i)}\big)}{(1+\delta^{(k)}\gamma^{(k)})\alpha^{(k)}\delta^{(k)}\sqrt{\phi^{(k)}(x_i)}K_1\big(\alpha^{(k)}\delta^{(k)}\sqrt{\phi^{(k)}(x_i)}\big) + (\alpha^{(k)})^2K_2\big(\alpha^{(k)}\delta^{(k)}\sqrt{\phi^{(k)}(x_i)}\big)} \qquad (17)$$

$$w_i^{(k)} = \frac{(1+\delta^{(k)}\gamma^{(k)})\alpha^{(k)}\delta^{(k)}\sqrt{\phi^{(k)}(x_i)}K_2\big(\alpha^{(k)}\delta^{(k)}\sqrt{\phi^{(k)}(x_i)}\big) + (\alpha^{(k)})^2K_3\big(\alpha^{(k)}\delta^{(k)}\sqrt{\phi^{(k)}(x_i)}\big)}{\alpha^{(k)}\delta^{(k)}\sqrt{\phi^{(k)}(x_i)}K_2\big(\alpha^{(k)}\delta^{(k)}\sqrt{\phi^{(k)}(x_i)}\big) + (1+\delta^{(k)}\gamma^{(k)})(\delta^{(k)})^2\phi^{(k)}(x_i)K_1\big(\alpha^{(k)}\delta^{(k)}\sqrt{\phi^{(k)}(x_i)}\big)} \qquad (18)$$

$$\hat{\gamma}^{(k+1)} = \frac{\delta^{(k)}\left[n - \frac{1}{(1+\delta^{(k)}\gamma^{(k)})}\sum_{i=1}^{n}\frac{1}{1+(1+\delta^{(k)}\gamma^{(k)})s_i^{(k)}}\right]}{\sum_{i=1}^{n}s_i^{(k)}} \qquad (19)$$

$$\hat{\beta}^{(k+1)} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})w_i^{(k)}}{n - \frac{1}{n}\sum_{i=1}^{n}s_i^{(k)}\sum_{i=1}^{n}w_i^{(k)}} \qquad (20)$$

$$\hat{\mu}^{(k+1)} = \bar{x} - \hat{\beta}^{(k+1)}\sum_{i=1}^{n}\frac{s_i^{(k)}}{n} \qquad (21)$$

$$\hat{\alpha}^{(k+1)} = [(\gamma^{(k+1)})^2 + (\beta^{(k+1)})^2]^{\frac{1}{2}} \qquad (22)$$

 

## 5 Application

Let $(P_t)$ denote the price process of a security at time $t$, in particular of a stock. In order to allow comparison of investments in different securities we shall investigate the rates of return defined by

$$X_t = \log P_t - \log P_{t-1}$$

In this section we consider three data sets for data analysis. The include: Range Resource Corporation (RRC), Shares of Chevron Corporation (CVX) and s&p500 index. The histogram for the weekly log-returns in Figure 1 for RRC illustrates that the data is negatively skewed and exhibiting heavy tails.The Q-Q plot shows that the normal distribution is not a good fit for the data especially at the tails. This is also similar for the other data sets.

Table 1 provides descriptive statistics for the return series in consideration. We observe that the data sets experience excess kurtosis indicates the leptokurtic behaviour of the returns. The log-returns has a distributions with relatively heavier tails than the normal distribution. The skewness indicates that the two tails of the returns behave differently.

**Table 1: Summary Statistics for the data sets.**

| dataset | Minimum | Standard.dev | skewness | exc.kurtosis | Maximum | Mean | N |
|---|---|---|---|---|---|---|---|
| RRC | -14.4465 | 2.824736 | -0.1886714 | 2.768252 | 13.9830 | 0.2333 | 702 |
| CVX | -13.76112 | 1.480436 | -1.297339 | 11.10113 | 6.71410 | 0.08711 | 702 |
| s&p500 | -8.722261 | 1.157893 | -0.7851156 | 6.408709 | 4.931805 | 0.006697 | 702 |

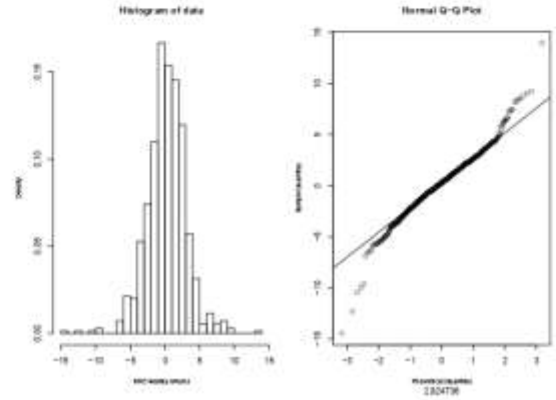


Figure 1: Histogram and Q-Q plot for RRC weekly log-returns

The table below gives the method of moment estimates of NIG for the three data sets. The estimates will be used as initial values for the EM-algorithm.

**Table 2: NIG method of moment estimates for the data sets.**

| dataset | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\delta}$ | $\hat{\mu}$ |
|---|---|---|---|---|
| RRC | 0.3722511 | -0.02456226 | 2.950864 | 0.4284473 |
| CVX | 0.4190067 | -0.1054991 | 0.8324058 | 0.3036691 |
| s&p500 | 0.6556607 | -0.1257455 | 0.8310044 | 0.1690855 |

The stopping criterion is when

$$\frac{l^{(k)} - l^{(k-1)}}{l^{(k)}} < tol \tag{23}$$

where tol is the tolerance level chosen; e.g $10^{-6}$ and $l^{(k)}$ as given in equation (11). We now wish to obtain the maximum likelihood parameter estimates of the data sets for the proposed model via the EM algorithm. Tables 3-5 illustrate monotonic convergence at different levels. The loglikelihood and AIC for each data sets are also provided.

**Table 3: Maximum likelihood estimates of the proposed model for RRC**

 

| Parameter | Starting Values | $EM(tol = 10^{-5})$ | $EM(tol = 10^{-6})$ | $EM(tol = 10^{-8})$ |
|---|---|---|---|---|
| $\hat{\alpha}$ | 0.3722511 | 0.5422119 | 0.5669625 | 0.5735228 |
| $\hat{\beta}$ | -0.02456226 | -0.04351162 | -0.04569187 | -0.04626743 |
| $\hat{\delta}$ | 2.950864 | 4.135342 | 4.279309 | 4.3181 |
| $\hat{\mu}$ | 0.4284473 | 0.561554 | 0.575256 | 0.5789067 |
| Loglikelihood | | -1698.1 | -1697.796 | -1697.745 |
| No. iteration | | 55 | 110 | 271 |
| AIC | | 3404.2 | 3403.592 | 3403.49 |

**Table 4: Maximum likelihood estimates of the proposed model for CVX**

| Parameter | Starting Values | $EM(tol = 10^{-5})$ | $EM(tol = 10^{-6})$ | $EM(tol = 10^{-8})$ |
|---|---|---|---|---|
| $\hat{\alpha}$ | 0.4190067 | 1.342608 | 1.382169 | 1.612872 |
| $\hat{\beta}$ | -0.1054991 | -0.3551768 | -0.3718762 | -0.4751398 |
| $\hat{\delta}$ | 0.8324058 | 2.491154 | 2.536825 | 2.805817 |
| $\hat{\mu}$ | 0.3036691 | 0.7494867 | 0.7756322 | 0.9361947 |
| Loglikelihood | | -1226.209 | -1226.186 | -1226.962 |
| No. iteration | | 48 | 55 | 490 |
| AIC | | 2460.418 | 2460.372 | 2453.912 |

**Table 5: Maximum likelihood estimates of the proposed model for s&p500 index**

| Parameter | Starting Values | $EM(tol = 10^{-5})$ | $EM(tol = 10^{-7})$ | $EM(tol = 10^{-8})$ |
|---|---|---|---|---|
| $\hat{\alpha}$ | 0.6556607 | 1.305925 | 1.644426 | 1.644773 |
| $\hat{\beta}$ | -0.1257455 | -0.1979512 | -0.2684969 | -0.2685751 |
| $\hat{\delta}$ | 0.8 | 1.633107 | 1.907611 | 1.90791 |
| $\hat{\mu}$ | 0.3036691 | 0.2371093 | 0.308193 | 0.3082736 |
| Loglikelihood | | -1047.278 | -1049.191 | -1049.194 |
| No. iteration | | 27 | 270 | 357 |
| AIC | | 2102.556 | 2106.382 | 2106.388 |

Figures 2, 3 and 4 show that the proposed models is a good fit the data sets.

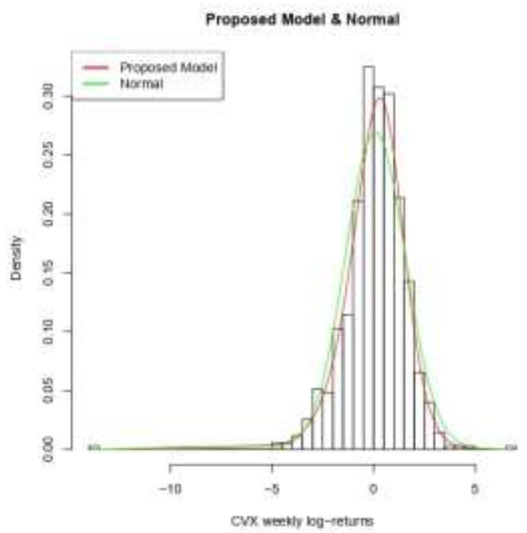


Figure 2: Fitting the proposed model to RRC log weekly returns

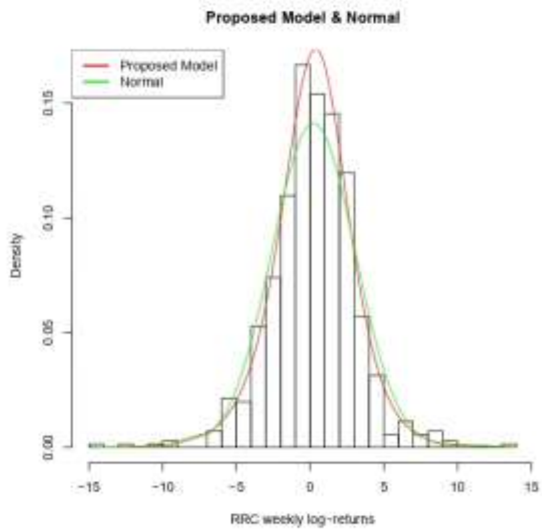


Figure 3: Fitting the proposed model to CVX log weekly returns

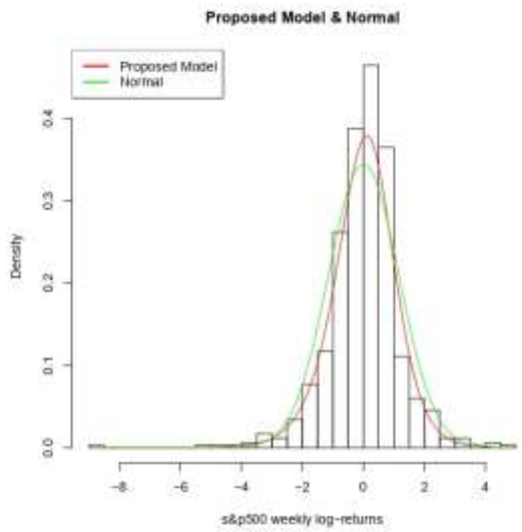


Figure 4: Fitting the proposed model to s&p500 index log weekly returns

## Remark:

Expressing the proposed model in terms of its components we have

$$f(x) = \frac{\delta^2}{1+\delta^2} \times NIG + \frac{1}{1+\delta^2} \times GHD(-\frac{3}{2}, \alpha, \delta, \beta, \mu) \quad (24)$$

Using the estimates we obtain the estimates of p for the data sets to be

**Table 6: estimates of p for the data sets.**

| dataset | $\hat{p}$ |
|---|---|
| RRC | 0.94910 |
| CVX | 0.88729 |
| s&p500 | 0.78449 |

The finite mixture for these data sets is more weighted to the NIG than the other special case of the GHD when $\lambda = -\frac{3}{2}$.

## 6 Conclusion

A finite mixture of two special cases of the Generalized Inverse Gaussian with indexes $-\frac{1}{2}$ and $-\frac{3}{2}$ have been shown to be Weighted Inverse Gaussian distributions. The mixture has been used as a mixing distribution for Normal Variance-Mean mixture to a Normal Weighted Inverse Gaussian Model.

Three data sets: Range Resource Corporation (RRC), Shares of Chevron Corporation (CVX) and s&p500 index for the period 3/01/2000 to 1/07/2013 with 702 observations have been used for data analysis. An iterative scheme has been presented for parameter estimation by the EM algorithm. The method of moment estimates for NIG have been used as initial values for the algorithm. The iterative scheme demonstrate a monotonic convergence. The model fits the data sets well.

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